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# Linear diffusion equations with singular absorption potentials and/or unbounded convective flows.

*Weights as boundary conditions: beyond Hardy-type inequalities*

David Gómez-Castro
*I. de Matemática Interdisciplinar, U. Complutense de Madrid*

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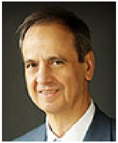
# Collaborators

Jesús Ildefonso Díaz
(Full Professor at UCM[1],
Real Academia de Ciencias)

Instituto de Matemática Interdisciplinar

Jean Michel Rakotoson
(Full Professor at U. Poitiers)



Roger Temam
(Full Professor at Indiana University[2],
French Academy of Sciences)

[1] Director of the Instituto de Matemática Interdisciplinar

[2] Director of the Institute for Scientific Computing and Applied Mathematics

# Introduction

The aim of this conference is to study

$$-\Delta u + \vec{b} \cdot \nabla u + V u = f$$

in the setting

$$V \in L^1_{loc}(\Omega), \qquad \vec{b} \in L^N(\Omega)^N, \qquad f\delta \in L^1(\Omega)$$

where

$$\delta(x) = d(x, \partial\Omega).$$

The results of this talk correspond to
J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, and R. Temam. "Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach". 2017

# Outline

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# Outline

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1. Motivations & Applications
   - Linearization of nonlinear problem
   - Shape differentiation
   - Schrödinger equation
   - Navier-Stokes equations

# Outline

# Linearization of singular and/or degenerate nonlinear equations

Consider

$$-\Delta\varphi(w) + \operatorname{div}\left(\vec{\phi}(w)\right) + g(w) = f(x) \qquad \text{in } \Omega \tag{1}$$

Context: stability of the associated parabolic or hyperbolic equations.
$\varphi \nearrow$. Take $\theta := \varphi(w)$ we get ($\vec{\psi} = \vec{\phi} \circ \varphi^{-1}, h = g \circ \varphi^{-1}$):

$$-\Delta\theta + \operatorname{div}\left(\vec{\psi}(\theta)\right) + h(\theta) = f(x) \qquad \text{in } \Omega, \tag{2}$$

Take $\theta_\infty(x)$ a solution of (2), s.t. $\theta = 0$ on $\partial\Omega$.
Then the "formal linearization" around the solution $\theta_\infty(x)$:

$$\vec{b}(x) := \vec{\psi}(\theta_\infty(x)) \qquad V(x) = h'(\theta_\infty(x)).$$

$\vec{\psi}'(r)$ and $h'(r)$ present a singularity at $r = 0$ (see[3])

[3] J. Hernández, F. J. Mancebo, and J. M. Vega. "On the linearization of some singular, nonlinear elliptic problems and applications". In: *Annales de l'IHP Analyse non linéaire.* Vol. 19. 6. 2002, pp. 777–813.

# Outline

# Application to shape differentiation

For $\Omega \subset \mathbb{R}^n$ smooth let us take $u(\Omega)$ the solution of the problem

$$\begin{cases} -\Delta u_\Omega + \beta(u_\Omega) = f, & \Omega \\ u_\Omega = 0, & \partial\Omega \end{cases}$$

Take $\Omega_0$ fixed and consider

$$\begin{array}{rcl} \{\text{deformation maps of } \Omega_0\} & \to & L^2(\mathbb{R}^n) \\ \Phi & \mapsto & u_{\Phi(\Omega_0)} \end{array}$$

Roughly speaking, $\Omega = \Phi(\Omega_0)$.

# Application to shape differentiation. Our motivation.

## Theorem (Díaz & G-C[a])

[a] J. I. Díaz and D. Gómez-Castro. "An Application of Shape Differentiation to the Effectiveness of a Steady State Reaction-Diffusion Problem Arising in Chemical Engineering". In: *Electronic Journal of Differential Equations* 22 (2015), pp. 31–45.

*Let* $\Omega_0$ *smooth,* $\beta \in W^{2,\infty}(\mathbb{R})$ *then the map*

$$\begin{aligned} F : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) &\to H_0^1(\Omega) \\ \theta &\mapsto u_{(I+\theta)(\Omega_0)} \circ (I + \theta) \end{aligned}$$

*is differentiable at* $\theta = 0$. *The directional derivative* $u'(\theta)$ *is the solution of the problem*

$$\begin{cases} -\Delta u' + \beta'(u_0) u' = 0, & \Omega_0 \\ u' + \theta \cdot \nabla u_0 = 0, & \partial\Omega_0 \end{cases} \tag{3}$$

*where* $u_0 = u_{\Omega_0} \in H_0^1(\Omega)$

# Application to shape differentiation: the non smooth case

Let following reaction term is very frequent in chemical catalysis [4]

$$g(s) = |s|^{q-1}s, \ 0 < q < 1, \qquad \text{Define } \beta(s) = g(1) - g(1-s) \tag{4}$$

Then the solution $u$ may develop a flat zone $N = \{u = 1\}$ ("dead core")[5]

A first estimate:

## Remark

$$V(x) = \beta'(u_0(x))) \sim d(x, N)^{-2} \text{ for } x \in \Omega \setminus N. \tag{5}$$

[4] The solution is $w = 1, \ \partial\Omega$ in Chemistry and $N = \{w = 0\}$. $u = 0$ on $\partial\Omega$ due to a change of variable.

[5] J. I. Díaz. *Nonlinear Partial Differential Equations and Free Boundaries.* London: Pitman, 1985.

# Outline

# The importance in the Schrödinger equation

If we consider the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\Delta \Psi + V \Psi$$

the separation $\Psi = u(x)T(t)$ yields

$$-\Delta u + V u = E u$$

**Remark**

Frequently in Physics the authors take

$$V(x) = d(x, \partial\Omega)^{-\alpha}.$$

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   - Schrödinger equation
   - Navier-Stokes equations

# The vorticity equation in fluid mechanics.

The stationary Navier-Stokes in 2D:

$$-\Delta\vec{b}+(\vec{b}\cdot\nabla)\vec{b}+\nabla p=\vec{F}$$

We get our problem taking the curl of the equation and setting

$$f=\vec{F}\cdot\vec{k},\qquad u=\operatorname{curl}\vec{b}\cdot\vec{k},$$

where $\vec{k}$ is the last element of the canonical basis in $\mathbb{R}^3$.
Nevertheless, as far as we know no satisfactory theory is available in the literature under the general condition that $\vec{F}\cdot\vec{k}\in L^1(\Omega;\delta)$.

# Outline

# Outline

2. Very weak solutions of $-\Delta u = f$
   - Uniqueness result
   - Continuous dependence
   - Existence result
   - Uniqueness result. Comparison principle
   - Kato's inequality for $u \in L^1_{loc}(\Omega)$
   - Uniqueness via Kato's inequality for $u \in W_0^{1,1}$
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# Very weak solutions of $-\Delta u = f$

Let $\Omega$ be smooth and bounded. We consider the problem

$$\begin{cases} -\Delta u = f & \Omega, \\ u = u_0 & \partial\Omega. \end{cases}$$

In this setting we have classically:

1. **classical solution.** $u \in C^2(\Omega) \cap C(\bar{\Omega})$
2. **weak solution.** Take a classical solution. Multiply the equation by $\varphi \in \mathcal{C}_2(\overline{\Omega}), \varphi = 0$ in $\partial\Omega$, integrate by parts:

$$\begin{aligned} \int_\Omega (-\Delta u)\varphi &= \int_\Omega f\varphi \\ -\int_\Omega \operatorname{div}(\varphi \nabla u) + \int_\Omega \nabla u \cdot \nabla \varphi &= \int_\Omega f\varphi \\ -\int_{\partial\Omega} (\varphi \nabla u) \cdot n + \int_\Omega \nabla u \cdot \nabla \varphi &= \int_\Omega f\varphi \\ \int_\Omega \nabla u \cdot \nabla \varphi &= \int_\Omega f\varphi. \end{aligned} \tag{6}$$

For $f \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$ we set

# Very weak solutions of $-\Delta u = f$

3. **very weak solution.**[6]: Integrating by part again we have

$$-\int_\Omega u \Delta \varphi = \int_\Omega f \varphi - \int_{\partial\Omega} u_0 \frac{\partial \varphi}{\partial n} \tag{7}$$

For $f \in L^1(\Omega, \delta), u_0 \in L^1(\partial\Omega)$ we define

$$\text{v.w.s.} \equiv \begin{cases} (7) \quad \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega) \\ u \in L^1(\Omega) \end{cases}$$

[6] H. Brézis. *Une équation non linéaire avec conditions aux limites dans $L^1$*. 1971.

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# Uniqueness result

The difference of two solutions $u = u_1 - u_2$ satisfies

$$-\int_\Omega u \Delta\varphi = 0, \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega).$$

Therefore $u = 0$.

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# Continuous dependence in the smooth case

Let $u$ the unique weak solution when $f, u_0$ are smooth. Then

$$\|u\|_{L^1(\Omega)} \le C(\|f\delta\|_{L^1(\Omega)} + \|u_0\|_{L^1(\partial\Omega)})$$

**Proof.**

The idea of the proof is simple.
For $\gamma$ a nondecreasing function $\gamma(0) = 0$, let $\gamma = \partial k$, $k(0) = 0$, $\kappa \ge 0$.
Let $\varphi \ge 0$. By considering a intelligent test function $\varphi\gamma(u)$ we obtain

$$\nabla u \cdot \nabla(\varphi\gamma(u)) = \nabla k(u) \cdot \nabla\varphi + \varphi\gamma'(u)|\nabla u|^2 \le \nabla k(u) \cdot \nabla\varphi.$$

Hence

$$\int_\Omega \nabla k(u)\nabla\varphi \le \int_\Omega f\varphi\gamma(u)$$

Let $\varphi \ge 0$ be given by

$$-\Delta\varphi = 1, \Omega \quad \varphi = 0 \partial\Omega.$$

Then

$$\int_{\partial\Omega} k(u_0)\frac{\partial\varphi}{\partial n} + \int_\Omega k(u)\underbrace{(-\Delta\varphi)}_{=1} \le \int_\Omega f\delta\underbrace{\delta^{-1}\varphi}_{\in L^\infty}\gamma(u)$$

$$\int_\Omega k(u) \le \|\varphi\|_{W^{1,\infty}}\left(\int_{\partial\Omega} k(u_0) + \int_\Omega \delta|f||\gamma(u)|\right)$$

As $\gamma \to \operatorname{sign}$ we have $k \to |\cdot|$.

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# Existence I

## Theorem

*Let $f\delta \in L^1(\Omega), u_0 \in L^1(\partial\Omega)$. Then there exists $u \in L^1(\Omega)$ such that*

$$-\int_\Omega u\Delta\varphi = \int_\Omega f\varphi - \int_{\partial\Omega} u_0 \frac{\partial\varphi}{\partial n} \quad \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega).$$

**Proof:**
Let $f_n \in \mathcal{C}(\bar{\Omega})$ and $u_{0n} \in \mathcal{C}(\partial\Omega)$ such that

$$f_n\delta \to f\delta \text{ in } L^1(\Omega), \quad u_{0n} \to u_0 \text{ in } L^1(\partial\Omega)$$

Then $(u_n)$ is a sequence of regular solutions such that

$$-\int_\Omega u_n\Delta\varphi = \int_\Omega f_n\varphi - \int_{\partial\Omega} u_{0n} \frac{\partial\varphi}{\partial n} \quad \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega).$$

# Existence II

By linearity $u_n - u_m$ is a solution of the problem $f_n - f_m$ and $u_{0n} - u_{0m}$ so

$$\|u_n - u_m\|_{L^1(\Omega)} \leq C(\|f_n\delta - f_m\delta\|_{L^1(\Omega)} + \|u_{n0} - u_{m0}\|_{L^1(\partial\Omega)})$$

Since $(f_n)$ and $(u_{0n})$ are Cauchy, so is $(u_n)$.
Hence, there exists $u \in L^1(\Omega)$ such that

$$u_n \to u \text{ in } L^1(\Omega)$$

For $\varphi \in W^{2,\infty}$ we have $\varphi, \Delta\varphi \in L^\infty(\Omega)$ and $\frac{\partial\varphi}{\partial n} \in L^\infty(\partial\Omega)$. Therefore

$$-\int_\Omega u\Delta\varphi = \int_\Omega f\varphi - \int_{\partial\Omega} u_0 \frac{\partial\varphi}{\partial n} \quad \forall\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega).$$

By convergence, $u$ also satisfies the continuous dependence equation. □

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# Uniqueness result. Comparison principle

Other than continuous dependence, the other trick to show uniqueness is the *comparison principle* (equivalently *maximum principle* or *monotonicity*) for smooth functions

$$\begin{cases} -\Delta u \le 0 & \Omega \\ u \le 0 & \partial\Omega \end{cases} \implies u \le 0 \quad \Omega$$

We can prove the uniqueness:

## Proof of the uniqueness via comparison. Smooth case.

Then, the difference of two solutions $u = u_1 - u_2$ satisfies this

$$u_1 \le u_2.$$

Then same holds for $u_2 - u_1$, therefore

$$u_2 \le u_1.$$

□

# Uniqueness via comparison principle

For smooth functions

$$\begin{cases} -\Delta u \le 0 & \Omega \\ u \le 0 & \partial\Omega \end{cases} \implies u \le 0 \quad \Omega$$

Question: does this work for very weak solutions?
Answer: YES

## Theorem

*Let* $u \in L^1(\Omega)$ *such that*

$$-\int_\Omega u \Delta \varphi \le 0 \quad \forall 0 \le \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$$

*Then*

$$u \le 0, \qquad \textit{a.e. } \Omega$$

# Uniqueness result. Comparison principle

$$-\int_\Omega u\Delta\varphi \le 0 \quad \forall 0 \le \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega) \implies u \le 0, \text{ a.e. } \Omega$$

## Idea of the proof for $u \in L^p(\Omega)$.

Take

$$-\Delta\varphi = \operatorname{sign}_+(u), \ \Omega \qquad \varphi = 0, \ \partial\Omega.$$

Which is $0 \le \varphi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$.

Take $0 \le \varphi_n \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ such that $\varphi_n \to \varphi$ in $W^{2,p'}(\Omega)$

Then

$$\int_\Omega u_+ \le 0.$$

Therefore $u_+ = 0$. Hence $u \le 0$. □

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# Kato's inequality for $u \in L^1(\Omega)$

## Definition

Let $u, f \in L^1_{loc}(\Omega)$. We say that $-\Delta u \le f$ in $\mathcal{D}'(\Omega)$ if

$$-\int_\Omega u \Delta \varphi \le \int_\Omega f \varphi \quad \forall 0 \le \varphi \in W^{2,\infty}_c(\Omega)$$

Notice that $d(\operatorname{supp} \varphi, \partial\Omega) > 0$. No information on the boundary conditions.

## Theorem (Kato's inequality[a])

[a] M. Marcus and L. Véron. *Nonlinear Second Order Elliptic Equations Involving Measures.* Vol. 22. De Gruyter, 2013.

*Assume that* $u, f \in L^1_{loc}(\Omega)$ *and* $-\Delta u \le f$ *in* $\mathcal{D}'(\Omega)$. *Then:*

1. $-\Delta |u| \le f \operatorname{sign} u$ *in* $\mathcal{D}'(\Omega)$.
2. $-\Delta u_+ \le f \operatorname{sign}_+ u$ *in* $\mathcal{D}'(\Omega)$.

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## Theorem (Maximum principle)

$$\begin{cases} -\Delta u \leq 0 \ \textit{in}\ \mathcal{D}'(\Omega) \\ u \in W_0^{1,1}(\Omega) \end{cases} \implies u \leq 0 \tag{8}$$

## Proof of the uniqueness result using Kato's inequality for $u \in W_0^{1,1}$.

Let $u_1, u_2 \in W_0^{1,1}(\Omega)$ be two solution. Let $u = u_1 - u_2$.
We have that $|u| \in W_0^{1,1}(\Omega)$ and $-\Delta |u| \leq 0 \cdot \operatorname{sign} u = 0$.
Therefore $|u| \leq 0$. □

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# Weights as boundary conditions. Beyond Hardy's inequality

## Theorem

*Hardy inequality Let* $u \in W^{1,p}(\Omega)$, $p > 1$. *Then*

$$\frac{u}{\delta} \in L^p(\Omega) \iff u \in W_0^{1,p}(\Omega).$$

*Let* $u \in W^{1,1}(\Omega)$. *Then*

$$\frac{u}{\delta} \in L^1(\Omega) \implies u \in W_0^{1,1}(\Omega).$$

*However* $\not\Longleftarrow$ .

QUESTION: Can we use $\frac{u}{\delta} \in L^1(\Omega)$ as boundary condition in $L^1(\Omega)$?

# QUESTION: Can $\frac{u}{\delta} \in L^1(\Omega)$ be b.c. in $L^1(\Omega)$?

ANSWER: YES

## Theorem

*Let $u$ and $r > 1$ be such that*

$$\begin{cases} -\Delta u \le 0 \text{ in } \mathcal{D}'(\Omega), \\ \dfrac{u}{\delta^r} \in L^1(\Omega). \end{cases}$$

*Then $u \le 0$.*

This guaranties uniqueness. This does not guaranty existence.
However, we will see there are problem in which $\frac{u}{\delta^r} \in L^1(\Omega)$ holds.

## Remark

The result holds for $r = 1$. Work in preparation.

[6] We write $-\Delta u \le f$ in $\mathcal{D}'(\Omega)$: $-\int_\Omega u \Delta \varphi \le \int_\Omega \varphi f$ for all $0 \le \varphi \in W_c^{2,\infty}(\Omega)$.

# An auxiliary function

Let $\psi \in \mathcal{C}^\infty(\mathbb{R})$ be $\nearrow$ s.t. $\psi(s) = \begin{cases} 1, & s \geq 1, \\ 0, & s \leq 0. \end{cases}$

Define, for $x \in \Omega$: $\varphi_\varepsilon(x) = \psi\left(\frac{\delta(x) - \varepsilon}{\varepsilon}\right) = \begin{cases} 0 & \text{if } \delta(x) \leq \varepsilon, \\ 1 & \text{if } \delta(x) \geq 2\varepsilon. \end{cases}$

Then:

1. $\operatorname{supp} \varphi_\varepsilon$ is compact
2. $\delta\varphi_\varepsilon \to \delta$ in $L^\infty(\Omega)$.
3. In the multiindex notation $\|D^\alpha \varphi_\varepsilon\|_\infty \leq C\varepsilon^{-|\alpha|}$
4. For $|\alpha| \geq 1$ we have $\operatorname{supp} D^\alpha \varphi_\varepsilon(x) \subset \{\varepsilon \leq \delta(x) \leq 2\varepsilon\}$.

## Theorem (Comparison Principle in $L^1(\Omega, \delta^{-r})$)

*Let $u$ and $r > 1$ be such that $-\Delta u \le 0$ in $\mathcal{D}'(\Omega)$ and $\frac{u}{\delta^r} \in L^1(\Omega)$. Then $u \le 0$.*

## Proof.

Let $\eta \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$. Then $\varphi_\varepsilon \eta \in W_c^{2,\infty}(\Omega)$. We have

$$\begin{aligned} \delta^r \Delta(\eta\varphi_\varepsilon) &= \delta\varphi_\varepsilon \Delta\eta + \delta^r \nabla\varphi_\varepsilon \cdot \nabla\eta + \delta^r \eta \Delta\varphi_\varepsilon \\ &= \delta\varphi_\varepsilon \Delta\eta + \delta^r \nabla\varphi_\varepsilon \cdot \nabla\eta + \frac{\eta}{\delta}\delta^{r+1}\Delta\varphi_\varepsilon \\ &\to \delta^r \Delta\eta \quad \text{in } L^\infty(\Omega). \end{aligned}$$

Then

$$0 \ge -\int_\Omega u\Delta(\varphi_\varepsilon\eta) = \int_\Omega \underbrace{\frac{u}{\delta^r}}_{L^1} \underbrace{\delta^r\Delta(\varphi_\varepsilon\eta)}_{L^\infty} \to -\int_\Omega \frac{u}{\delta^r}\delta^r\Delta\eta = -\int_\Omega u\Delta\eta.$$

□

# Last comments

Regularity of very weak was studied in:

J. I. Díaz and J. M. Rakotoson. "On the differentiability of very weak solutions with right hand side data integrable with respect to the distance to the boundary". In: *Journal of Functional Analysis* 257.3 (2009), pp. 807–831.

# Outline

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3. Very weak solutions of $-\Delta u + Vu = f, V \geq 0$
   - The case $V \leq C\delta^{-2}$. Hardy's inequality
   - Existence for $V \in L^1_{loc}$
   - Uniqueness for $V \in L^1(\Omega, \delta)$
   - Uniqueness when $V \geq c\delta^{-r}, r > 2$

# The comfortable setting

In the section we aim to work on the following PDE

$$\begin{cases} -\Delta u + Vu = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{9}$$

when $V \in L^1_{loc}(\Omega)$. It is well known

$$\left.\begin{matrix} f \in L^2(\Omega), \\ V \in L^\infty(\Omega) \end{matrix}\right\} \implies \text{Existence and Uniqueness of } u \in H^1_0(\Omega)$$

We apply Lax-Milgram to the following problem

$$\begin{cases} \text{Find } u \in H^1_0(\Omega) \text{ such that } \forall \varphi \in H^1_0(\Omega) \\ \displaystyle\int_\Omega \nabla u \nabla \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi \end{cases}$$

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# The "easy" cases: $V \leq c\delta^{-2}$

Due to Hardy's inequality

$$u\delta^{-1} \in L^2(\Omega), \qquad \forall u \in H_0^1(\Omega) \tag{10}$$

Hence

$$\begin{aligned} H_0^1(\Omega) \times H_0^1(\Omega) &\to \mathbb{R} \qquad (11) \\ (u,\varphi) &\mapsto \int_\Omega V u \varphi = \int_\Omega \underbrace{(V\delta^2)}_{L^\infty} \underbrace{(u\delta^{-1})}_{L^2} \underbrace{(\varphi\delta^{-1})}_{L^2} \qquad (12) \end{aligned}$$

is bilinear continuous. Hence we can apply Lax-Milgram.

It is known[7] that $V \leq c\delta^{-\alpha}$, $\alpha \in (0,1)$

$$u \in W_0^{1,1}(\Omega), \qquad \|\nabla u\|_{L^{\frac{N}{N-1+\alpha},\infty}(\Omega)} \leq \|f\|_{L^1(\Omega,\delta^\alpha)} \tag{13}$$

[7] J. I. Díaz and J. M. Rakotoson. "On very weak solutions of Semi-linear elliptic equations in the framework of weighted spaces with respect to the distance to the boundary". In: *Discrete and Continuous Dynamical Systems* 27.3 (2010), pp. 1037–1058.

Let $\mu(x)$ be a function. We define

$$L^p(\Omega, \mu) = \left\{ f \text{ measurable } : \int_\Omega |f|^p \mu < +\infty \right\}$$

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# Existence for $V \in L^1_{loc}$

## Theorem (Díaz & Rakotoson[a])

[a] J. I. Díaz and J. M. Rakotoson. "On very weak solutions of Semi-linear elliptic equations in the framework of weighted spaces with respect to the distance to the boundary". In: *Discrete and Continuous Dynamical Systems* 27.3 (2010), pp. 1037–1058.

*If* $V \in L^1_{loc}(\Omega)$ *and* $f \in L^1(\Omega, \delta)$ *then there exists* $u \in L^1(\Omega)$ *solution of*

$$\begin{cases} Vu \in L^1(\Omega, \delta), \text{ and } \forall \varphi \in W^{2,\infty}_c(\Omega), \\ -\displaystyle\int_\Omega u \Delta \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \end{cases}$$

*If* $V, f \in L^1(\Omega, \delta)$ *then there exists* $u \in L^1(\Omega)$ *solution of*

$$\begin{cases} Vu \in L^1(\Omega, \delta), \text{ and } \forall \varphi \in W^{2,\infty}(\Omega) \cap W^{1,\infty}_0(\Omega), \\ -\displaystyle\int_\Omega u \Delta \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \end{cases}$$

# Some remarks

1. Extra regularity is known:

$$u \in L^{N',\infty} \cap W^{1,q}(\Omega, \delta).$$

2. Some extra bounds are known for at least one of the solutions:
   1. $\|u\|_{L^{N',\infty}} \leq \|f\delta\|_{L^1}$
3. The case $V\delta \in L^1(\Omega)$ has uniqueness in a direct way.
4. If $V = \delta^{-\alpha}$ then $V\delta \in L^1(\Omega) \iff \alpha < 2$.

# Existence for $V \in L^1_{loc}$

The argument is the following. Let $T_k(s) = \begin{cases} s, & |s \leq k \\ k \text{ sign } s, & |s| > k. \end{cases}$

$$V_k = T_k(V), \qquad f_k = T_k(f).$$

Let $u_k$ be the weak solution of

$$-\Delta u_k + V_k u_k = f_k, \Omega, \qquad u_k = 0, \partial\Omega$$

Then, applying continuous dependence:

$$u_k \xrightarrow{L^1(\Omega)} u, \qquad \text{as } k \to +\infty.$$

Besides $V u \delta \in L^1(\Omega)$. Through a Dunford-Pettis argument:

1. $V_k u_k \delta \rightharpoonup V u \delta$ in $L^1_{loc}(\Omega)$.
2. If $V\delta \in L^1(\Omega)$ then $V_k u_k \delta \rightharpoonup V u \delta$ in $L^1(\Omega)$

# Outline

3. Very weak solutions of $-\Delta u + Vu = f, V \geq 0$
   - The case $V \leq C\delta^{-2}$. Hardy's inequality
   - Existence for $V \in L^1_{loc}$
   - Uniqueness for $V \in L^1(\Omega, \delta)$
   - Uniqueness when $V \geq c\delta^{-r}, r > 2$

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# Proof of uniqueness for $V \in L^1(\Omega, \delta)$

**Step 1.** how that $\varphi \geq 0$

$$-\int_\Omega |u| \Delta \varphi + \int_\Omega V |u| \varphi \leq \int_\Omega f \varphi$$

## Proof.

Let

$$-\Delta \varphi = 1, \Omega \quad \varphi = 0 \partial \Omega.$$

By considering a intelligent test function $\varphi \gamma(u_n)$ we obtain

$$\int_\Omega \nabla k(u_n) \nabla \varphi + \int_\Omega V_k u_k \gamma(u_n) \leq \int_\Omega f \varphi \gamma(u_n)$$

$$-\int_\Omega k(u_n) \Delta \varphi + \int_\Omega V_n u_n \gamma(u_n) \leq \int_\Omega f \varphi \gamma(u_n)$$

We pass to the limit in $n$. As $\gamma \to \operatorname{sign}$ we have $k \to |\cdot|$. □

# Uniqueness for $V \in L^1(\Omega, \delta)$

**Step 2.** Take two solutions $u_1, u_2$ and $u = u_1 - u_2$.

$$-\int_\Omega |u|\Delta\varphi + \int_\Omega V|u|\varphi \le 0$$

**Step 3.** Then we define $\varphi_1 \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$:

$$-\Delta\varphi_1 = \lambda_1\varphi_1, \Omega \qquad \varphi_1 = 0, \partial\Omega.$$

Hence

$$\int_\Omega |u| \underbrace{(\lambda_1 + V)\varphi_1}_{>0} \le 0$$

Therefore $u = 0$.

# Outline

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# Uniqueness when $V \geq c\delta^{-r}, r > 2$ I

## Theorem

*Let* $0 \leq V \in L^1_{loc}(\Omega)$ *and* $u$ *be such that*

$$\begin{cases} -\displaystyle\int_\Omega u\Delta\varphi + \int_\Omega Vu\varphi = 0 \quad \forall \varphi \in W_c^{2,\infty}(\Omega), \\ Vu \in L^1_{loc}(\Omega), \quad \dfrac{u}{\delta^s} \in L^1(\Omega), s > 1. \end{cases}$$

*Then* $u = 0$*.*

## Proof.

We write $-\Delta u = -Vu \in L^1_{loc}(\Omega)$. By Kato's inequality

$$-\Delta|u| \leq -Vu \operatorname{sign}(u) = -V|u| \leq 0 \text{ in } \mathcal{D}'(\Omega).$$

Since $|u| \in L^1(\Omega, \delta^{-s})$ we have that $|u| \leq 0$. Hence $u = 0$. □

# Uniqueness when $V \geq c\delta^{-r}, r > 2$ II

If $V \geq C\delta^{-r}$ with $C > 0$. We have

$$0 \leq |u|\delta^{1-r} \leq \frac{1}{C} V|u|\delta \in L^1(\Omega).$$

Therefore:

## Corollary ([a])

[a] J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, and R. Temam. "Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach". 2017.

*Let $C\delta^{-r} \leq V \in L^1_{loc}(\Omega)$, with $C > 0, r > 2$, and $f\delta \in L^1(\Omega)$. Then, there exists a unique v.w.s.*

# Outline

# Outline

4. Very weak solutions $-\Delta u + \vec{b} \cdot \nabla u + Vu = f$
   - Conditions on $\vec{b}$
   - Definition of very weak solution
   - Study of the dual problem
   - Lorentz spaces
   - Regularity results for the dual problem
   - Existence of very weak solutions
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# Conditions on $\vec{b}$

We will consider the conditions

$$\begin{cases} \operatorname{div} \vec{b} = 0 & \Omega, \\ \vec{b} \cdot \vec{n} = 0 & \partial\Omega. \end{cases} \tag{14}$$

In the following sense:

$$\int_\Omega \varphi(\vec{b} \cdot \nabla u) = - \int_\Omega u(\vec{b} \cdot \nabla \varphi), \qquad \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega).$$

## Remark

For smooth functions $\operatorname{div}(\varphi \vec{b}) = \varphi \operatorname{div} \vec{b} + \vec{b} \cdot \nabla \varphi$

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# Definition of very weak solution

## Definition

Let $f$ be in $L^1(\Omega;\delta)$ and $\vec{b} \in L^{N,1}(\Omega)^N$ with $\operatorname{div}(\vec{b}) = 0$ in $\mathcal{D}'(\Omega)$, $\vec{b} \cdot \vec{n} = 0$ on $\partial\Omega$, $V$ measurable and non negative function. A very weak solution $u$ is a function $u \in L^{N',\infty}(\Omega)$ such that

$$V u \in L^1(\Omega;\delta) \text{ and } \int_\Omega u\big[-\Delta\phi - \vec{b}\cdot\nabla\phi + V\phi\big]dx = \int_\Omega f\phi\,dx, \qquad (15)$$

for all $\phi \in C^2(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$, if $V \in L^1(\Omega;\delta)$, or for all $\phi \in C_c^2(\Omega)$ if $V \in L^1_{loc}(\Omega)$.

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# From monotonicity methods to the study of dual solutions

The methods we used in the previous equations are no longer easy. Nonetheless we write

$$\int_u u(-\Delta\psi - \vec{b}\cdot\nabla\psi + V\psi) = \int_u f\psi.$$

Let

$$L^*\psi = -\Delta\psi - \vec{b}\cdot\nabla\psi + V\psi.$$

When $\vec{b} = 0$ it was easy to find functions $\psi$ very regular such that

$$L^*\psi = 1 \qquad L^*\psi = \operatorname{sign}_+ u.$$

This kind of arguments is harder now.

To obtain this results this we do regularity escalation in Lorentz spaces.

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# Lorentz spaces

Let $f : \Omega \to \mathbb{R}$ be measurable.
The distribution function of $f$: $\mu(t) = |\{x \in u : |f(x)| > t\}|$.
The decreasing rearrangement of $f$: $f^*(s) = \sup\{t \geq 0 : \mu(t) > s\}$

Lorentz defined the following spaces[8][9]:
Given $0 < p, q \leq \infty$ define

$$\|f\|_{(p,q)} = \begin{cases} \left( \int_0^\infty \left( t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & q < +\infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & q = +\infty. \end{cases}$$

and $L^{(p,q)}(\Omega) = \{f \text{ measurable in } \Omega : \|f\|_{(p,q)} < +\infty\}$

[8] G. Lorentz. "Some new functional spaces". In: *Annals of Mathematics* 51.2 (1950), pp. 37–55.

[9] G. Lorentz. "On the theory of spaces $\Lambda$". In: *Pacific Journal of Mathematics* 1.3 (1951), pp. 411–430.

# An alternative definition of Lorentz spaces

Let $1 \leqslant p \leqslant +\infty,\ 1 \leqslant q \leqslant +\infty$ . Let $u \in L^0(\Omega)$. We define

$$||u||_{p,q} = \begin{cases} \left[\int_{\Omega_*} \left[t^{\frac{1}{p}} |u|_{**}(t)\right]^q \frac{dt}{t}\right]^{\frac{1}{q}} & q < +\infty, \\ \sup_{0<t\leqslant|\Omega|} t^{\frac{1}{p}} |u|_{**}(t) & q = +\infty \end{cases} \qquad |u|_{**}(t) = \frac{1}{t}\int_0^t |u|_*(\sigma) d\sigma.$$

We define $L^{p,q}(\Omega) = \{f \text{ measurable in } \Omega : \|u\|_{p,q} < +\infty\}$.

## Proposition (Corollary 1.4.1 in[a])

[a] J.-M. Rakotoson. *Réarrangement Relatif.* Vol. 64. Mathématiques et Applications. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008.

*Let* $1 < p \leqslant +\infty, 1 \leqslant q \leqslant +\infty$. *Then*

$$L^{p,q}(\Omega) = L^{(p,q)}(\Omega)$$

*with equivalent quasi-norms.*

# Properties of Lorentz spaces

The functionals $\| \cdot \|_{L^{p,q}}$ do not, in general, satisfy the triangle inequality. However, $L^{p,q}$ is a quasi-Banach space.

The following properties are known[10]:

1. If $0 < p \leq \infty$ and $0 < q < r \leq +\infty$ then $L^{(p,q)} \subset L^{(p,r)}$.
2. $L^{(p,p)} = L^p$ for all $p \geq 1$.
3. Let $1 \leq p, q < \infty$. Then $(L^{(p,q)}(\Omega))' = L^{(p',q')}(\Omega)$.
4. If $q < r < p$ then $L^{(p,\infty)}(\Omega) \cap L^{(q,\infty)}(\Omega) \subset L^r(\Omega)$ (even for $\Omega$ unbounded)
5. In a bounded domain, if $r < p$ then $L^{(p,\infty)}(\Omega) \subset L^r(\Omega)$

[10] L. Grafakos. *Classical Fourier Analysis*. Vol. 249. Graduate Texts in Mathematics. New York, NY: Springer New York, 2009, pp. 1–1013.

# Lorentz spaces. Inclusion diagram

We can write the diagram of inclusions for $1 \le q \le r < p < +\infty$

$$\begin{tikzcd}
 & & & L^{(\infty,\infty)} = L^{\infty} \arrow[dlll, hook'] \\
L^{(p,1)} \arrow[r, hook] \arrow[d, hook] & L^{(p,q)} \arrow[rr, hook] \arrow[d, hook] & & L^{(p,\infty)} = \text{weak } L^{p} \arrow[dl, hook'] \\
L^{(r,1)} \arrow[r, hook] & L^{(r,q)} \arrow[r, hook] & L^{(r,r)} = L^{r} \arrow[r, hook] & L^{(r,\infty)}.
\end{tikzcd}$$

# Sobolev spaces of Lorentz spaces

We can define the Sobolev spaces associated to this spaces. In general, let $X \subset L^0$. We define the

$$W^m X = \{ f \in X : \forall \alpha \text{ such that } |\alpha| \le m \text{ we have } D^\alpha f \in X \} \tag{16}$$

where $D^\alpha$ is the generalized derivative of order $\alpha$ (in the multiindex notation).

# The highlights

## Remark

In order to read the following slides it is enough to keep in mind the following

$$1 \leq p < q < r \implies L^r \subset L^{q,1} \subset L^q \subset L^{q,\infty} \subset L^p$$

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# Regularity results for the dual problem

## Proposition

*Let* $T \in H^{-1}(\Omega)$ *(dual of* $H_0^1(\Omega)$*),* $\vec{b}$ *satisfying (5) and let* $0 \leq V \in L^0(\Omega)$ *.*
*Define* $W = \left\{ \varphi \in H_0^1(\Omega) : V\varphi^2 \in L^1(\Omega) \right\}$*, and let* $W'$ *denotes its dual. Then, there exists a unique* $\phi \in H_0^1(\Omega)$*, with* $V\phi^2 \in L^1(\Omega)$*, such that*

$$(\mathcal{P})_{V,T} \qquad -\Delta\phi - \vec{b}\cdot\nabla\phi + V\phi = T \text{ in } W'. \tag{17}$$

*Moreover,*

$$||\phi||_{H_0^1(\Omega)} \leq c||T||_{H^{-1}(\Omega)},$$
$$\left(\int_\Omega V\phi^2 dx\right)^{\frac{1}{2}} \leqslant c||T||_{H^{-1}(\Omega)},$$

*If furthermore* $V \in L^1_{loc}(\Omega$*, then the equation (17) holds in the sense of distributions in* $\mathcal{D}'(\Omega)$

# Regularity results for the dual problem

## Proposition (Approximation by bounded potentials)

*Let $T \in H^{-1}(\Omega)$, $\vec{b}$ and $V$. Then, the sequence $\phi_k \in H_0^1(\Omega)$ of solutions of the problems*

$$(\mathcal{P})_{V_k,T} : \int_\Omega \nabla\phi_k \cdot \nabla\psi dx - \int_\Omega \vec{b}\nabla\phi_k \phi dx + \int_\Omega V_k \phi_k \psi dx = \langle T, \psi \rangle, \quad \forall\, \psi \in H_0^1(\Omega),$$

*converges to $\phi$ strongly in $H_0^1(\Omega)$, where $\phi$ is the unique solution of $(\mathcal{P})_{V,T}$ found in Proposition 2.*

## Proposition

*Under the same assumptions as for Proposition 2 (with $\lambda = 0$), if $T \geqslant 0,\ T \in L^1(\Omega) \cap H^{-1}(\Omega)$ then $\phi \geqslant 0$.*

## Regularity results for the dual problem

### Proposition ($L^\infty$-estimates)

*Let $\phi$ be the solution of (17) when $T \in L^{\frac{N}{2},1}(\Omega)$, $V \geqslant 0$. Then $\phi \in L^\infty(\Omega)$ and there exists a constant $K_N(\Omega)$ independent of $\vec{b}$ and $V$ such that*

$$||\phi||_{L^\infty(\Omega)} \leqslant K_N(\Omega)||T||_{L^{\frac{N}{2},1}(\Omega)}.$$

### Proposition

*Let $N \geqslant 2$, and let $\phi$ be a solution of (17) when*

$$T = -\operatorname{div}(\vec{F}), \quad \vec{F} \in L_V = \begin{cases} L^{N,1}(\Omega)^N & N \geqslant 3, \\ L^{2+\varepsilon}(\Omega)^2 & N = 2. \end{cases}$$

*Then $\phi \in L^\infty(\Omega)$ and there exists a constant $K_N(\Omega) > 0$ independent of $\vec{b}$ and $V$ such that*

$$||\phi||_{L^\infty(\Omega)} \leqslant K_N(\Omega)||\vec{F}||_{L_V}.$$

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# Regularity results for the dual problem

## Proposition ($W^1L^{p,q}$-estimate)

*Let $N \geqslant 2$. Assume that there exists $p > N$ and $q \in [1, +\infty]$, such that*

$$\begin{cases} \vec{b} \in L^{p,q}(\Omega)^N & V \geqslant 0,\ V \in L^{r,q}(\Omega),\ r = \dfrac{Np}{N+p}, \\ T = -\operatorname{div}(\vec{F}) & \text{with } \vec{F} \in L^{p,q}(\Omega)^N. \end{cases}$$

*Then, the unique solution $\phi$ of the equation (17) belongs to $W^1L^{p,q}(\Omega)$. Moreover, there exists a constant $K_{pq} > 0$ independent of $\vec{b}$ such that :*

$$||\nabla\phi||_{L^{p,q}(\Omega)} \leqslant K_{pq}\left(1 + ||\vec{b}||_{L^{p,q}} + ||V||_{L^{r,q}}\right)||F||_{L^{p,q}(\Omega)^N}.$$

## Proposition

*Let $\vec{b}$ and $\vec{F}$ be in $L^{p,\infty}(\Omega)^N$ for some $p > N$. Then, the solution of (17) satisfies*

$$\phi \in C^{0,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{p}.$$

# Regularity results for the dual problem

## Proposition ($W^2L^{p,q}(\Omega)$ regularity for $p>N$)

*Let $\phi$ be the solution of (17) when $T \in L^{p,q}(\Omega),\ p > N,\ q \in [1,+\infty]$. Assume, furthermore, that $\vec{b} \in L^{p,q}(\Omega)^N$ and $V \in L^{p,q}(\Omega)$. Then*

$$\phi \in W^2L^{p,q}(\Omega).$$

*Moreover, there exist constants $c_{\varepsilon_0},\ K_{pqN} > 0$ such that*

$$||\phi||_{W^2L^{p,q}(\Omega)} \leqslant \frac{K_{pqN}c_{\varepsilon_0}(1+||V||_{L^{p,q}}+||\vec{b}||_{L^{p,q}(\Omega)})}{1-K^s_{pq}\varepsilon_0||\vec{b}||_{L^{p,q}(\Omega)}}||T||_{L^{p,q}(\Omega)}.$$

# Regularity results for the dual problem

The case where $p = N$ can also be treated in the same way provided that the norm of $\vec{b}$ in $L^{N,1}(\Omega)$ is small enough in the sense that

$$||\vec{b}||_{L^{N,1}(\Omega)} \leqslant \theta K_{N1}^{s0} \text{ for some } \theta \in [\,0,1\,[, \quad (18)$$

$$K_{N1}^{s0} = K_{N1}^{s} \sup_{\phi \in H_0^1(\Omega) \cap W^2 L^{N,1}(\Omega)} \frac{||\nabla \phi||_\infty}{||\phi||_{W^2 L^{N,1}}}. \quad (19)$$

## Proposition (regularity in $W^2 L^{N,1}(\Omega)$)

*Let $\phi$ be the solution of (17) when $T \in L^{N,1}(\Omega)$, $V \in L^{N,1}(\Omega)$. Assume that $\vec{b}$ satisfies relation (18). Then $\phi \in W^2 L^{N,1}(\Omega)$. Moreover, there exists a constant $K_N'(\Omega)$ (independent of $\vec{b}$) such that*

$$||\phi||_{W^2 L^{N,1}(\Omega)} \leqslant \frac{K_N'(\Omega)(1 + ||V||_{L^{N,1}})}{1 - K_{N1}^{s0}||\vec{b}||_{L^{N,1}}} ||T||_{L^{N,1}(\Omega)}.$$

# Regularity results for the dual problem

Here we want only to consider the space $\Lambda = (L^N(\operatorname{Log} L)^{\frac{\beta}{N}})^N$ for $\beta > N-1$.
Indeed this space is included in $L^{N,1}(\Omega)$ and contains $L^p(\Omega)$ for all $p > N$.

## Theorem (regularity in $W^2L^N(\Omega)$)

*Let $T$ and $V$ be in $L^N(\Omega)$, $\vec{b} \in \Lambda$, $\operatorname{div}(\vec{b}) = 0$ and $\vec{b}\cdot\vec{n} = 0$ on $\partial\Omega$. Then the unique solution $\phi$ of (17) belongs to $W^2L^N(\Omega)$ and choosing $\varepsilon > 0$ such that $\varepsilon||\vec{b}||_\Lambda \leqslant \frac{1}{2}$, there exists a constant $K_\varepsilon > 0$ such that*

$$||\phi||_{W^2L^N(\Omega)} \leqslant \frac{K_\varepsilon(1 + ||\vec{b}||_\Lambda + ||V||_{L^N})}{1 - \varepsilon||\vec{b}||_\Lambda}||T||_{L^N(\Omega)}.$$

# Outline

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# Existence of very weak solutions

## Theorem

*Let* $f \in L^1(\Omega;\delta)$.
*Let* $\vec{b}$ *be in* $L^{p,1}(\Omega)^N$ *with* $\mathrm{div}(\vec{b}) = 0$ *in* $\mathcal{D}'(\Omega)$, $\vec{b} \cdot \vec{n} = 0$ *on* $\partial\Omega$.

*Furthermore, assume that either* $p > N$ *or* $p = N$ *and* $||\vec{b}||_{L^{N,1}} < K_{N1}^{s0}$ *(see (18)).*

*Then, there exists a very weak solution* $u$ *of (15).*

*Furthermore, If* $V \in L^{p,1}(\Omega)$, *then the solution is unique.*

# Existence of very weak solutions. Sketch of proof

1. Take $f \geq 0$. $-\Delta u_j + \vec{b}_j \cdot \nabla u_j + V_j u_j = f_j$
2. $\|u_j\|_{L^{N',\infty}} \leq C \|\delta f_j\|_{L^1(\Omega)}$
   1. $E$ measurable. Take $-\Delta \phi_j - \vec{b}_j \cdot \nabla \phi_j = \chi_E$ .
   2. $\int_E \omega_j \leq C|E|^{\frac{1}{N}} \|f_j \delta\|_{L^1}$
   3. Hardy Littlewood inequality.
3. $u_j \rightharpoonup u$ in $L^{N',\infty}(\Omega)$. We deduce strong convergence from boundedness in $W^{1,q}(\Omega, \delta)$ for $q > 1$ small.
4. Check that $\|V \omega \delta\| \leq C(1 + \|\vec{b}\|_{L^{N,1}}) \|f \delta\|_{L^1}$ .
5. $\vec{b}_j u_j \to bu$ in $L^1(\Omega)^N$, using pointwise convergence and Vitali's condition.
6. Dunford-Pettis argument $V_j u_j \delta \rightharpoonup V$.
7. Pass to the limit in the equation.
8. If $V \in L^{p,1}$ take $\phi \in W^2 L^{N,1}$ solution of $-\Delta \phi - \vec{b} \cdot \nabla \phi + V \phi = \text{sign}\,(u)$

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# Outline

# A comparison principle with transport I

## Theorem (Comparison principle)

*Let* $\overline{u}$ *be in* $L^1(\Omega;\delta^{-r}) \cap W^{1,1}_{\mathrm{loc}}(\Omega),\ r > 1$. *Let* $\overline{u} \in L^{N',\infty}(\Omega)$ *and* $\vec{b} \in L^{p,1}(\Omega)$ *with* $p > N$ *or* $p = N$ *with a small norm. Assume that*

$$L\overline{u} \doteq -\Delta\overline{u} + \operatorname{div}(\vec{b}\,\overline{u}) \leqslant 0 \text{ in } \mathcal{D}'(\Omega).$$

*and* $L\overline{u} \in L^1_{loc}(\Omega)$.
*Then*

$$\overline{u} \leqslant 0 \text{ in } \Omega.$$

# A comparison principle with transport II

## Corollary (Variant of Kato's inequality)

*Let* $\overline{u}$ *be in* $W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^{N',\infty}(\Omega)$, $\overline{u} \in L^1(\Omega; \delta^{-r})$ *for* $r > 1$ *and*

$$\vec{b} \in L^{N,1}(\Omega)^N \text{ with } \operatorname{div}(\vec{b}) = 0,\ \vec{b} \cdot \vec{n} = 0.$$

*Assume furthermore that* $L\overline{u} = -\Delta \overline{u} + \operatorname{div}(\vec{b}\overline{u})$ *is in* $L^1(\Omega; \delta)$. *Then for all* $\phi \in C^2(\overline{\Omega}),\ \phi = 0$ *on* $\partial\Omega$, $\phi \geqslant 0$ *one has*

1. $\displaystyle\int_\Omega \overline{u}_+ L^* \phi dx \leqslant \int_\Omega \phi \operatorname{sign}_+(\overline{u}) L(\overline{u}) dx,$
2. $\displaystyle\int_\Omega |u| L^* \phi dx \leqslant \int_\Omega \phi \operatorname{sign}(\overline{u}) L(\overline{u}) dx,$

*where* $L^*\phi = -\Delta\phi - \vec{b} \cdot \nabla \phi = -\Delta \phi - \operatorname{div}(\vec{b}\phi)$.

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# Uniqueness when $V \geq c\delta^{-r}, r > 2$

## Theorem

*Assume that* $0 \leq V \in L^1_{loc}(u)$, *and such that*

$$\exists\, c > 0,\ V(x) \geqslant c\delta(x)^{-r},\ \text{in a neighborhood } U \text{ of the boundary, with } r > 2.$$

*Then, the v.w.s.* $u$ *found is unique.*

## Sketch of Proof.

We have that

$$|u|\delta^{1-r} \leq \frac{1}{c} V|u|\delta \in L^1(u).$$

Again take $u = u_1 - u_2$ and

$$L^*|u| \leq -V|u|$$

Therefore $u = 0$. □

Thank you for your attention

# References I


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