Fractional Schrödinger equation with singular data and potential

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Aim

We study equations of the form

$$\begin{cases} Lu + Vu = f & \Omega, \\ u = 0 & \partial\Omega \ (\text{resp. } \Omega^c), \end{cases} \tag{P_V}$$

where L is an integro-differential operator, e.g. $(-\Delta)^s$,
posed on a bounded domain Ω of $\mathbb{R}^n$, where $n \geq 3$ and $0 < s < 1$.

V (the potential) is a nonnegative Borel measurable function. It may be singular.

f is some function or measure. We aim to study the singular cases.
The results correspond to the publications

J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez. "The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach". *Nonlinear Analysis* (2018), pp. 1–36. arXiv: 1804.08398

Structure of the talk

1. Fractional operators
2. The Laplace equation ($V = 0$)
3. Problem (P_V) for $V \in L^1$
4. Data measures
5. Potentials singular at interior points
6. Singular potentials at the boundary: $V \in L^1_{loc}$. The RFL

Outline

1. Fractional operators

2. The Laplace equation ($V = 0$)

3. Problem (P_V) for $V \in L^1$

4. Data measures

5. Potentials singular at interior points

6. Singular potentials at the boundary: $V \in L^1_{loc}$. The RFL

Fractional Laplacian in $\mathbb{R}^n$

Let $0 < s < 1$. The following definitions are equivalent:

- Fourier transform $\mathcal{F}$, with simbol $|\xi|^{2s}$:

$$(-\Delta)^s u = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}[u]] \tag{1a}$$

- As a singular integral

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-(y)}{|x-y|^{n+2s}}dy \tag{1b}$$

- Through the heat semigroup

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)}\int_0^{+\infty} (e^{\Delta t}u(x) - u(x))\frac{dt}{t^{1+s}} \tag{1c}$$

Probabilistically, where $-\Delta$ correspond to Brownian motion, $(-\Delta)^s$ corresponds to Levy flights.
Clearly $(-\Delta)^1 = -\Delta$ and $(-\Delta)^0 = Id$.

The fractional Laplacians on bounded domains

The following non-equivalent definitions on bounded domains are common:

- The **Restricted** Fractional Laplacian (RFL):

$$(-\Delta)^s_{\mathrm{RFL}} u(x) = c_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \tag{2}$$

  where u is extended by 0 outside Ω

- The **Spectral** Fractional Laplacian (SFL)

$$(-\Delta)^s_{\mathrm{SFL}} u(x) = \frac{1}{\Gamma(-s)}\int_0^{+\infty} \left(e^{t\Delta}u(x) - u(x)\right)\frac{dt}{t^{1+s}} \tag{3}$$

  This corresponds to computing the spectral decomposition and defining

$$(-\Delta)^s_{\mathrm{SFL}} u(x) = \sum_{i=1}^{+\infty} u_i \lambda_i^s \varphi_i.$$

- The **Censored** (or Regional) Fractional Laplacian (CFL), for $1/2 < s < 1$:

$$(-\Delta)^s_{\mathrm{CFL}} u(x) = c_{n,s}\,\mathrm{P.V.}\int_{\Omega} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \tag{4}$$

Outline

The Laplace problem

Homogeneous Dirichlet problem

$$\begin{cases} Lu = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \qquad L = -\Delta, (-\Delta)^s_{\mathrm{SFL}}, (-\Delta)^s_{\mathrm{CFL}} \tag{P$_0$}$$

$$\begin{cases} Lu = f & \Omega \\ u = 0 & \Omega^c \end{cases} \qquad L = (-\Delta)^s_{\mathrm{RFL}} \tag{P$_0$}$$

For the operators above and $f \in C_c^\infty(\Omega)$, there exists a unique classical solution $u \in \mathcal{C}(\overline{\Omega})$ (i.e. satisfying the problem pointwise).

This solution can be represented as

$$u(x) = \int_\Omega \mathbb{G}(x,y) f(y) dy. \tag{G}$$

We define the solution operator

$$\mathrm{G} : f \longmapsto u. \tag{5}$$

Existence of solutions in energy spaces. Weak solutions

The operators in the example satisfy a theory similar to the classical.

- **RFL:** We can write a weak formulation, for $f \in L^2$ find $u \in H^s_0(\Omega)$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega f\varphi, \qquad \forall \varphi \in H^s_0(\Omega)$$

- **CFL:** We can write a weak formulation, for $f \in L^2$ find $u \in H^s_0(\Omega)$ such that

$$\int_{\Omega}\int_{\Omega} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega f\varphi, \qquad \forall \varphi \in H^s_0(\Omega)$$

- **SFL:** Write $f \in L^2$ in eigen-decomposition

$$f = \sum_{i=1}^{+\infty} f_i \varphi_i \qquad \longmapsto \qquad u = \sum_{i=1}^{+\infty} f_i \lambda_i^{-s} \varphi_i .$$

Problems for the RFL and CFL can be solved by minimization of energy. Also by using the standard Lax-Milgram theorem.
For fractional sobolev spaces: Di Nezza, Palatucci, and Valdinoci 2012.

Kernel representation

We assume that L is an operator such that, for $f \in L^\infty(\Omega)$

$$\mathrm{G}(f)(x) = \int_\Omega \mathbb{G}(x,y) f(y) dy. \tag{G}$$

such that G satisfies properties:

(i) G is symmetric and self-adjoint in the sense that

$$\mathbb{G}(x,y) = \mathbb{G}(y,x). \tag{G1}$$

(ii) We assume $n \geq 3$ and we have the estimate

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right)^\gamma . \tag{G2}$$

The examples covered by our theory

The elliptic problem has been widely studied, specially for data $f \in L^\infty(\Omega)$. Also, two-sided kernel estimates are known:

- **The classical Laplacian $-\Delta$:**
  (G2) holds with $s = 1$ and $\gamma = 1$.
- **Restricted Fractional Laplacian $(-\Delta)^s_{\mathrm{RFL}}$:**
  (G2) holds with $0 < s < 1$ and $\gamma = s$. See Chen and Song 1998.
- **Spectral Fractional Laplacian $(-\Delta)^s_{\mathrm{SFL}}$:**
  (G2) holds with $0 < s < 1$ and $\gamma = 1$.
  See Bonforte, Figalli, and Vázquez 2018.
- **Censored Fractional Laplacian $(-\Delta)^s_{\mathrm{CFL}}$:**
  (G2) holds with $\frac{1}{2} < s < 1$ and $\gamma = 2s - 1$. See Chen, Kim, and Song 2009.

Weak and weak-dual formulations

Since L is self-adjoint, we can define very weak solutions

$$\int_\Omega uL\varphi = \int_\Omega f\varphi \qquad \forall \varphi \in ? \tag{6}$$

Different operators require different sets of test functions. Some authors use weak solutions for the RFL Chen and Véron 2014; Díaz, Gómez-Castro, and Vázquez 2018.

Letting $\varphi = \mathrm{G}(\psi)$ we can simply write

$$\int_\Omega u\psi = \int_\Omega f\mathrm{G}(\psi) \qquad \forall \psi \in L^\infty(\Omega) \tag{P$_0$-WD}$$

This kind of solutions are known as *weak-dual solutions*.
Since L is self-adjoint, G is self-adjoint and so

$$\int_\Omega \mathrm{G}(f)\psi = \int_\Omega f\mathrm{G}(\psi), \qquad \forall f, \psi \in L^\infty(\Omega) \tag{7}$$

Hence, classical solutions for $f \in L^\infty(\Omega)$ are weak-dual solutions.

Estimates for some simple functions

Theorem

Assume (G2). *Then, we have that:*

- *For* $K \Subset \Omega$: $\mathrm{G}(\chi_K) \asymp \delta^\gamma$
- *For* $A \subset \Omega$: *we have* $|\mathrm{G}(\chi_A)| \le C|A|^\beta$, *for any* $\beta < \frac{2s}{n}$, *where* $C = C(\beta)$.

From now on we assume (G1)–(G2).

Uniqueness of weak-dual solutions

Theorem

Let $f \in L^1(\Omega)$. Then there exists, at most, one solution $u \in L^1(\Omega)$ of

$$\int_\Omega u\psi = \int_\Omega f \mathrm{G}(\psi) \qquad \forall \psi \in L^\infty(\Omega) \tag{P_0-WD}$$

Proof.

Let $u_1, u_2 \in L^1(\Omega)$ be two solutions. Taking $\psi = \mathrm{sign}(u_1 - u_2) \in L^\infty(\Omega)$ we deduce

$$\int_\Omega |u_1 - u_2| = \int_\Omega (u_1 - u_2)\, \mathrm{sign}(u_1 - u_2) = \int_\Omega (f - f)\mathrm{G}(\mathrm{sign}(u_1 - u_2)) = 0. \tag{8}$$

□

Remark

From the positive cone to the whole space

Lemma

Let $p, q > 1$ be two normed function spaces, and let $T : L^p \to L^q$ be linear and continuous. If

$$\|T(f)\|_{L^q} \leq C\|f\|_{L^p}, \qquad \forall 0 \leq f \in X$$

Then the same holds for any $f \in X$.

Proof.

Let $f \in X$. We split $f = f_+ + f_-$. Then

$$\|T(f)\|_q = \|T(f_+) - T(f_-)\|_q \leq \|T(f_+)\|_q + \|T(f_-)\|_q \leq C\|f_+\|_p + C\|f_-\|_p = \tag{9}$$

□

Regularization

Theorem

$$f \in L^p(\Omega) \implies \mathrm{G}(f) \in L^q(\Omega) \qquad \forall 1 \le q < Q(p) = \frac{n}{n-2s}p.$$

Furthermore $\mathrm{G} : L^p(\Omega) \to L^q(\Omega)$ *is continuous.*

Proof.

The $L^1(\Omega)$ and $L^\infty(\Omega)$ result follow by direct computation.
The intermediate case by Riesz-Thorin lemma. □

Dunford-Pettis property

The aim of this section is to prove that

Theorem

We have that, for any $0 < \beta < \frac{2s}{n}$

$$\int_A |\mathrm{G}(f)| \le C|A|^\beta \|f\|_{L^1(\Omega)}, \qquad \forall f \in L^1(\Omega). \tag{10}$$

for some $C > 0$.

Hence, if $f_n \in L^1(\Omega)$ *is a bounded sequence, then* $\mathrm{G}(f_n)$ *is uniformly integrable.*

In particular, there exists a weakly convergent subsequence $\mathrm{G}(f_{n_k}) \rightharpoonup u$ *in* $L^1(\Omega)$.

Extension to L^1

Through duality and approximation we prove that:

Theorem

Let G *satisfy* (G1)–(G2)*. Then, there exists an extension*

$$\mathrm{G} : L^1(\Omega) \to L^1(\Omega). \tag{11}$$

which is linear and continuous.
Furthermore, this extension is unique and self-adjoint.
The function $u = \mathrm{G}(f)$ *is the unique function such that* $u \in L^1(\Omega)$ *and*

$$\int_\Omega u\psi = \int_\Omega \mathrm{G}(\psi) f. \tag{P$_0$-WD}$$

This solution can be represented as

$$u(x) = \int_\Omega \mathbb{G}(x,y) f(y) dy. \tag{12}$$

Optimal set of data f

Due to the duality, it is easy to check that

$$\int_\Omega \mathrm{G}(f) = \int_\Omega f \mathrm{G}(1).$$

Therefore

$$\mathrm{G} : L^1(\Omega, \mathrm{G}(1)) \to L^1(\Omega).$$

On the other hand, for $K \Subset \Omega$,

$$\int_K \mathrm{G}(f) = \int_\Omega f \mathrm{G}(\chi_K).$$

For this operators $\mathrm{G}(\chi_K) \asymp \delta^\gamma$, and hence

$$\mathrm{G} : L^1(\Omega, \delta^\gamma) \to L^1_{loc}(\Omega).$$

For $f \geq 0$ we have, due to (G2)

$$\mathrm{G}(f)(x) = \int_\Omega \mathbb{G}(x,y) f(y) dy \geq c\delta(x)^\gamma \int_\Omega f(y)\delta(y)^\gamma$$

If $f\delta^\gamma \notin L^1 \implies \mathrm{G}(f) \equiv +\infty$.

Outline

Back to the Schrödinger problem

We can rewrite

$$\begin{cases} Lu + Vu = f & \Omega, \\ u = 0 & \partial\Omega \ (\text{resp. } \Omega^c), \end{cases} \tag{P_V}$$

We write the problem as a fixed point:

$$u = \mathrm{G}(f - Vu) \tag{P_V-D}$$

We call this *dual formulation*.

This is equivalent to the *weak-dual formulation*

$$\int_\Omega u\psi + \int_\Omega Vu\mathrm{G}(\psi) = \int_\Omega f\mathrm{G}(\psi) \qquad \forall \psi \in L^\infty(\Omega). \tag{P_V-WD}$$

Existence for $(f, V) \in L^1(\Omega) \times L^\infty_+(\Omega)$

Here we show the following

Theorem

Let $f \in L^1(\Omega)$ and $V \in L^\infty_+(\Omega)$. Then, there exists a solution u of (P_V-D) *and it satisfies*

$$|u| \le \mathrm{G}(|f|)$$

Furthermore,

$$f \ge 0 \implies u \ge 0.$$

We have

$$|\mathrm{G}_V(f)| \le \mathrm{G}(|f|). \tag{13}$$

Proof for $f \geq 0$.

We construct the following sequence. $u_0 = 0$, $u_1 = \mathrm{G}(f) \geq 0$,

$$u_2 = \mathrm{G}\left(\left(f - Vu_1\right)_+\right), \qquad u_i = \mathrm{G}(f - Vu_{i-1}), \qquad i > 2. \tag{14}$$

Step a. We prove that

$$u_0 \leq u_2 \leq u_3 \leq u_1. \tag{15}$$

Step b. We show, by induction, that

$$u_{2i} \leq u_{2i+2} \leq u_{2i+3} \leq u_{2i+1}, \qquad \forall i \geq 0. \tag{16}$$

Step c. By the monotone convergence theorem $u_{2i} \nearrow \underline{u}$ in $L^1(\Omega)$ where $u_{2i+1} \searrow \overline{u}$ in $L^1(\Omega)$. We have

$$\overline{u} = \mathrm{G}(f - V\underline{u}), \qquad \underline{u} = \mathrm{G}(f - V\overline{u}). \tag{17}$$

Therefore $u = \frac{1}{2}(\underline{u} + \overline{u})$ is a solution of (P_V-D). □

Uniqueness for $(f, V) \in L^1(\Omega) \times L^\infty_+(\Omega)$

In order to prove uniqueness we to assume

(iii) Furthermore, we need positivity in the sense that

$$\int_\Omega f \mathrm{G}(f) \ge 0 \qquad \forall f \in L^2(\Omega) \tag{G3}$$

This is true for the examples.

Theorem

$V \in L^\infty(\Omega)$. There exists at most one solution $u \in L^1(\Omega)$ of (P_V-D).

Proof.

The difference of two solutions $u = u_1 - u_2 \in L^1(\Omega)$ satisfies $u = -\mathrm{G}(Vu)$. We have that $Vu^2 = -Vu\mathrm{G}(Vu) \in L^2(\Omega)$. We deduce

$$0 \le \int_\Omega Vu^2 = -\int_\Omega Vu\mathrm{G}(Vu) \underbrace{\le}_{\text{(G3)}} 0.$$

Hence $Vu^2 = 0$ so $u = -\mathrm{G}(0) = 0$. □

Solution operator

Corollary

Let $V \in L^\infty_+(\Omega)$*. We consider the solution operator*

$$\mathrm{G}_V : f \in L^1(\Omega) \mapsto u \in L^1(\Omega)\,,$$

where u *is the unique solution of* $u = \mathrm{G}(f - Vu)$*. It is well-defined, linear and continuous.*

Corollary

We have, for any $0 \le \beta < 2s/n$,

$$\int_A |\mathrm{G}_V(f)| \le C|A|^\beta \|f\|_{L^1(\Omega)}, \qquad \forall f \in L^1(\Omega).$$

where $C = C(\beta)$*.*

In particular, $f_n \in L^1(\Omega)$ *bounded* $\implies$ $\mathrm{G}(f_n)$ *uniformly integrable.*

Notion of solution when $V \notin L^\infty_+$

In order for (P_V-D) to be well defined, we need to require something extra. We extend the definition by setting that Vu in the admissible class

$$\begin{cases} u = \mathrm{G}(f - Vu) \\ Vu \in L^1(\Omega, \delta^\gamma) \end{cases} \tag{P_V-D}$$

Theorem

For any dual solution we have that:

$$\int_\Omega |u| \le C \int_\Omega |f| \qquad \mathit{and} \qquad \int_\Omega V|u|\delta^\gamma \le C \int_\Omega |f|\delta^\gamma. \tag{18}$$

where C does not depend on V, f.

Proof in the case $f \ge 0$.

Setting $\psi = 1$ we get, since

$$\int_\Omega u + \int_\Omega Vu\mathrm{G}(1) \le \|\mathrm{G}(1)\|_\infty \int_\Omega f.$$

Taking $\psi = \chi_K$ for any $K \Subset \Omega$, since $\mathrm{G}(\chi_K) \asymp \delta^\gamma$:

$$\int_K u + \int_\Omega Vu\delta^\gamma \le C \int_\Omega f\delta^\gamma.$$

□

Uniqueness for general $V \geq 0$

Theorem

Assume $|\{V = +\infty\}| = 0$.
There exists, at most, one solution $u \in L^1(\Omega)$ *of* $(\mathrm{P}_V\text{-D})$.

Proof.

Let $u_1, u_2 \in L^1(\Omega)$ be two solutions. Then $u = u_1 - u_2$ satisfies $u = -\mathrm{G}(Vu)$.
For $k \in \mathbb{N}$ we define $V_k = V \wedge k \in L^\infty_+(\Omega)$.
We write

$$u = \mathrm{G}((V_k - V)u - V_k u) = \mathrm{G}(f_k - V_k u) \tag{19}$$

where $f_k = (V_k - V)u \in L^1(\Omega)$.
Hence, due to Theorem 3.2, u is the unique solution of $u + \mathrm{G}(V_k u) = \mathrm{G}(f_k)$ and

$$\|u\|_{L^1(\Omega)} \leq C\|f_k\|_{L^1(\Omega)}. \tag{20}$$

On the other hand, we have that

$$|f_k| = |(V - V_k)u| \leq |V - V_k||u| \leq V|u| \in L^1(\Omega).$$

Then

$$V_k \to V \text{ a.e.} \implies f_k = (V_k - V)u \to 0 \text{ a.e.} \overset{DCT}{\implies} f_k \to 0 \text{ in } L^1(\Omega).$$

Existence for $(f, V) \in L^1(\Omega) \times L^1_+(\Omega)$

Theorem

If $(f, V) \in L^1(\Omega) \times L^1_+(\Omega)$, there exists a solution.

Lemma (Monotonicity)

If $V_1 \le V_2$ and $f_1 \ge f_2$ then $\mathrm{G}_{V_1}(f_1) \ge \mathrm{G}_{V_2}(f_2)$.

Proof for $f \geq 0$.

We define

$$V_k = V \wedge k, \qquad f_m = f \wedge m.$$

We define $u_{k,m} = \mathrm{G}_{V_k}(f_m) \in L^\infty(\Omega)$. Let $U_m = \mathrm{G}(f_m) \in L^\infty(\Omega)$. Clearly $u_{k,m} \leq U_m$

Step a. $k \to +\infty$. $V_k \nearrow \implies 0 \leq u_{k,m} \searrow \overset{\mathrm{MCT}}{\implies} u_{k,m} \to u_m \text{ in } L^1(\Omega)$. On the other hand

$$\begin{cases} V_k u_{k,m} \leq V U_m \in L^1(\Omega) \\ V_k u_{k,m} \to V u_m \text{ a.e. } \Omega \end{cases} \overset{\mathrm{DCT}}{\implies} V_k u_{k,m} \to V u_m \text{ in } L^1(\Omega). \tag{21}$$

Hence

$$u_m = \lim_k u_{k,m} = \lim_k \mathrm{G}(f_m - V_k u_{k,m}) = \mathrm{G}(f_m - V u_m) \tag{22}$$

and u_m is the solution corresponding to (f_m, V). □

Proof (cont.)

We define

$$V_k = V \wedge k, \qquad f_m = f \wedge m. \tag{23}$$

We define $u_{k,m} = \mathrm{G}_{V_k}(f_m) \in L^\infty(\Omega)$. Let $U_m = \mathrm{G}(f_m) \in L^\infty(\Omega)$.

Step a. $k \to +\infty$**.** As $V_k \nearrow V$ we have $u_{k,m} \searrow u_m = \mathrm{G}(f_m - Vu_m)$.

Step b. $m \to +\infty$**.** Since $f_m \nearrow \implies u_m \nearrow$.

$$\int_\Omega u_m \le C \overset{\mathrm{MCT}}{\implies} u_m \to u \text{ in } L^1(\Omega).$$

Analogously $Vu_m\delta^\gamma \nearrow Vu\delta^\gamma$ in $L^1(\Omega)$.
Furthermore $u_m = \mathrm{G}(f_m - V_m u_m) \to \mathrm{G}(f - Vu)$. □

Outline

Regularization and measure data

(iv) Assume G is regularizing in the sense that

$$\mathrm{G} : L^\infty(\Omega) \to \mathcal{C}(\overline{\Omega}). \tag{G4}$$

Then

Theorem

Let G satisfy (G1)–(G4). Then, there exists a extension

$$\mathrm{G} : \mathcal{M}(\Omega) \to L^1(\Omega).$$

which is linear and continuous. Furthermore, this extension is unique and self-adjoint. The function $u = \mathrm{G}(\mu)$ is the unique function such that $u \in L^1(\Omega)$ and

$$\int_\Omega u\psi = \int_\Omega \mathrm{G}(\psi)\mathrm{d}\mu.$$

This solution can be represented as

$$u(x) = \int_\Omega \mathbb{G}(x,y)d\mu(y).$$

The examples

- $-\Delta$: (G4) is a classical result. See, e.g., Evans 1998; Gilbarg and Trudinger 2001.
- $(-\Delta)^s_{\mathrm{RFL}}$: (G4) is proven via Hörmander theory. See, e.g. Grubb 2015; Ros-Oton and Serra 2014.
- $(-\Delta)^s_{\mathrm{SFL}}$: (G4) can be found in Caffarelli and Stinga 2016.

Outline

CSOLAs

We prove the following, where δ_x is the Dirac measure at $x \in \Omega$,

Theorem

Assume that $V \geq 0$ such that

$$V : \Omega \to [0, +\infty] \text{ is measurable and } L^\infty(\Omega \setminus B_\rho(S)) \text{ for all } \rho > 0, \tag{V1}$$

for a finite set S, and let $\mu \geq 0$ be a nonnegative Radon measure. Then, there exist an integrable function $\underline{u} \geq 0$ and constants $(\alpha_\mu^x)_{x \in S} \in \mathbb{R}$ such that:

i) $\mathrm{G}_{V_k}(\mu) = u_k \searrow \underline{u}$ *in* $L^1(\Omega)$

ii) $V_k u_k \to V \underline{u}$ *in* $L^1(\Omega \setminus B_\rho(S), \delta^\gamma)$ *for any* $\rho > 0$

iii) $V_k u_k \rightharpoonup V \underline{u} + \sum_{x \in S} \alpha_\mu^x \delta_x$ *weakly in* $\mathcal{M}(\Omega, \delta^\gamma)$.

iv) *The limit satisfies the equation, for the reduced measure* $\mu_r = \mu - \sum_{x \in S} \alpha_\mu^x \delta_x$.

$$\underline{u} + \mathrm{G}(V \underline{u}) = \mathrm{G}(\mu_r), \tag{24}$$

If a solution exists it is the limit

It is easy to prove, that :

Lemma

If there exists $u = \mathrm{G}(\mu - Vu)$, *then* $u = \underline{u}$.

Proof.

Assume $f \geq 0$. Let $w_k = u_k - u$. Then

$$w_k = \mathrm{G}(\mu - V_k u_k) - \mathrm{G}(\mu - Vu) \tag{25}$$

$$= \mathrm{G}(Vu - V_k u_k) \tag{26}$$

$$= \mathrm{G}(\underbrace{(V - V_k)u}_{f_k} - V_k w_k) \tag{27}$$

Hence $\|w_k\|_{L^1} \leq C\|(V - V_k)u\|_{L^1}$.
Via MCT $(V - V_k)u \searrow 0$ in $L^1(\Omega)$. Hence

$$w_k \to 0 \text{ in } L^1(\Omega). \tag{28}$$

Data $\mu = \delta_x$

In this case we have

$$(\delta_x)_r = (1 - \alpha^x)\delta_x \qquad (29)$$

There are only two options

1. $(1 - \alpha^x) \neq 0$. Then $\frac{\underline{u}}{1-\alpha^x}$ is a solution. Then, it is easy to prove that $\alpha^x = 0$.
2. $1 - \alpha^x = 0$. By uniqueness, then $\underline{u} = 0$.

There exists a set

$$Z = \{x \in \Omega : \text{there is no solution of } (\mathrm{P}_V) \text{ with data } \delta_x\} \subset S. \qquad (30)$$

Necessary and sufficient condition for existence

Through scaling, one can characterize

$$\mu_r = \mu - \sum_{x \in Z} \mu(\{x\})\delta_x. \tag{31}$$

where, we recall

$$Z = \{x \in \Omega : \text{there is no solution of } (\mathrm{P}_V) \text{ with data } \delta_x\}. \tag{32}$$

One can prove that

Theorem

Let $\mu \in \mathcal{M}(\Omega)$.
There exists a dual solution of (P_V) with data $\mu \iff |\mu|(Z) = 0$.

This is compatible for the results for the usual laplacian given in Orsina and Ponce 2018.

The CSOLA operator

Let us define

$$\begin{aligned} \widetilde{\mathrm{G}}_V : \mathcal{M}(\Omega) &\longrightarrow L^1(\Omega) \\ \mu &\longmapsto \mathrm{G}_V(\mu_r). \end{aligned}$$

Then $\widetilde{\mathrm{G}}_V$ is the unique self-adjoint extension of G_V to $\mathcal{M}(\Omega)$. This operator admits a kernel representation

$$\begin{aligned} \mathrm{G}_V(f)(x) &= \langle \mathrm{G}_V(f), \delta_x \rangle = \langle \widetilde{\mathrm{G}}_V(f), \delta_x \rangle = \langle f, \mathrm{G}_V(\delta_x) \rangle \\ &= \int_\Omega \widetilde{\mathrm{G}}_V(\delta_x)(y) f(y) dy. \end{aligned}$$

hence

$$\mathbb{G}_V(x,y) = \widetilde{\mathrm{G}}_V(\delta_x)(y). \tag{33}$$

The maximum principle

Notice that, amongst the previous computation, we showed that, for $x \in \Omega$,

$$\mathrm{G}_V(f)(x) = \langle f, .\widetilde{\mathrm{G}}_V(\delta_x) \rangle$$

We also recall that

$$\widetilde{\mathrm{G}}_V(\delta_x) = \mathrm{G}_V((\delta_x)_r) = 0 \qquad \forall x \in Z.$$

But then,

$$\mathrm{G}(f)(x) = 0 \qquad \forall x \in Z.$$

No maximum principle on Z.
For the usual laplacian: Orsina and Ponce 2018.

Characterization of Z

Theorem

Assume (V1). *Then*

$$x \notin Z \iff \int_{B_\rho(x)} \frac{V(y)}{|x-y|^{n-2s}} dy < +\infty. \textit{ for some } \rho > 0. \quad (34)$$

In particular, $Z \subset S$. *The second condition is* $V\mathrm{G}(\delta_0) \in L^1(B_\rho(x))$.

Proof of $\Longleftarrow$.

We may take $x = 0$ for convenience. Let $U = \mathrm{G}(\delta_0) \in L^1(\Omega)$.
(i) Assume first $VU \in L^1(\Omega)$. Approximate by $u_k = \mathrm{G}_{V_k}(\delta_0)$. We have

$$\begin{aligned} V_k u_k \le VU \in L^1(B_\rho(x)) \overset{\mathrm{DCT}}{\Longrightarrow} V_k u_k \to V\underline{u} \in L^1(B_\rho(x)) &\implies \alpha_\mu^0 = 0 \\ &\implies (\delta_0)_r = \delta_0 \\ &\implies \text{There exists a solution for } \mu = \delta_0 \\ &\implies 0 \notin Z \end{aligned}$$

Outline

In this section we discuss the results in

J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez. "The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach". *Nonlinear Analysis* (2018), pp. 1–36. arXiv: 1804.08398

We study the case of the RFL. The aim is to extend previous results for the classical case given in

J. I. Díaz, D. Gómez-Castro, J.-M. Rakotoson, and R. Temam. "Linear diffusion with singular absorption potential and/or unbounded convective flow: The weighted space approach". *Discrete and Continuous Dynamical Systems* 38.2 (2018), pp. 509–546. arXiv: 1710.07048

J. I. Díaz, D. Gómez-Castro, and J.-M. Rakotoson. "Existence and uniqueness of solutions of Schrödinger type stationary equations with very singular potentials without prescribing boundary conditions and some applications". *Differential Equations & Applications* 10.1 (2018), pp. 47–74. arXiv: 1710.06679

Optimal set of data f for the RFL

For the RFL we have $s = \gamma$ and

$$\mathrm{G}(1) \asymp \delta^s.$$

Hence

$$\mathrm{G} : L^1(\Omega, \delta^s) \to L^1(\Omega).$$

and

$$0 \le f \notin L^1(\Omega, \delta^s) \implies \mathrm{G}(f) \equiv +\infty.$$

Weak dual solutions

In order for

$$u = \mathrm{G}(f - Vu) \qquad (\mathrm{P}_V\text{-D})$$

we will require

$$Vu\delta^s \in L^1(\Omega). \qquad (\mathrm{P}_V\text{-D-b})$$

Let us look at this in W-D:

$$\int_\Omega u\psi + \int_\Omega Vu\mathrm{G}(\psi) = \int_\Omega f\mathrm{G}(\psi).$$

Consider $f \ge 0$. Then $u \ge 0$ hence

$$\int_\Omega |u| + \int_\Omega V|u|\mathrm{G}(1) = \int_\Omega f\mathrm{G}(1).$$

Therefore, since $\mathrm{G}(1) \asymp \delta^s$

$$\int_\Omega Vu\delta^s \le \int_\Omega f\delta^s.$$

For f changing sign

$$\int V|u|\delta^s \le \int |f|\delta^s.$$

Approximation

Assume $V \in L^1_{loc}$.

Going back to $u_{k,m} = \mathrm{G}_{V_k}(f_m)$ with $f_m \geq 0$:

As $V_k \nearrow V \implies 0 \leq u_{k,m} \searrow u_m$ in L^1.

We have, taking $\psi = 1$.

$$\int_\Omega u_{k,m} + \int_\Omega V_k u_{k,m} \delta^s \leq C \int_\Omega f_m \delta^s. \tag{35}$$

Also

$$V_k u_{k,m} \leq \underbrace{V}_{L^1_{loc}} \underbrace{\mathrm{G}(f_m)}_{L^\infty} \in L^1_{loc}(\Omega)$$

This implies, together with the a.e. convergence,

$$V_k u_{k,m} \to V u_m \in L^1_{loc}(\Omega). \tag{36}$$

However, this does not seem to implies L^1 convergence. We have

$$\int_\Omega V u_m \delta^s \leq \int_\Omega f_m \delta^s \tag{37}$$

Difficulties of the case $V \in L^1_{loc}$

We showed

$$V_k u_{k,m} \to V u_m \in L^1_{loc}(\Omega). \tag{36}$$

We would need, in the weak formulation,

$$\mathrm{G}(\psi) \in L^\infty_c(\Omega).$$

This does not happen for $\psi \in L^\infty$. We go back to the very weak formulation

$$\int_\Omega u_{k,m} (-\Delta)^s_{\mathrm{RFL}} \varphi + \int_\Omega V_k u_{k,m} \varphi = \int_\Omega f_m \varphi$$

for all $\varphi \in \mathbb{X}^s = \{\varphi \in C^s(\mathbb{R}^n) : (-\Delta)^s_{\mathrm{RFL}} \varphi \in L^\infty(\Omega)\}$.

If we multiply by a cut-off function $\eta_\varepsilon \in C^\infty_c(\Omega)$ we get, formally

$$\int_\Omega u_{k,m} (-\Delta)^s_{\mathrm{RFL}} (\eta_\varepsilon \varphi) + \int_\Omega V_k u_{k,m} \eta_\varepsilon \varphi = \int_\Omega f_m \eta_\varepsilon \varphi$$

Now, we do have

$$\int V_k u_{k,m} \eta_\varepsilon \varphi \to \int V u_m \eta_\varepsilon \varphi$$

Approximation of test functions

Let η be a $C^2(\mathbb{R})$ function such that $0 \le \eta \le 1$ and

$$\eta(t) = \begin{cases} 0 & t \le 0, \\ 1 & t \ge 2. \end{cases} \tag{38}$$

We define the functions

$$\eta_\varepsilon(x) = \eta\left(\frac{\varphi_1(x) - \varepsilon^s}{\varepsilon^s}\right). \tag{39}$$

where φ_1 is the first eigenfunction of $(-\Delta)^s_{\mathrm{RFL}}$. Notice that $\varphi_1 \asymp \delta^s$. We prove the following approximation result:

Lemma

For $\varphi \in \mathbb{X}^s$ we have that $\eta_\varepsilon\varphi \in \mathbb{X}^s \cap C_c(\Omega)$ and

$$\delta^s(-\Delta)^s(\varphi\eta_\varepsilon) \rightharpoonup \delta^s(-\Delta)^s\varphi \tag{40}$$

$$\frac{\varphi\eta_\varepsilon}{\delta^s} \rightharpoonup \frac{\varphi}{\delta^s} \tag{41}$$

Approximation of test functions

There exists a sequence

Lemma

For $\varphi \in \mathbb{X}^s$ we have that $\eta_\varepsilon \varphi \in \mathbb{X}^s \cap C_c(\Omega)$ and

$$\delta^s(-\Delta)^s(\varphi\eta_\varepsilon) \rightharpoonup \delta^s(-\Delta)^s\varphi \tag{40}$$

$$\frac{\varphi\eta_\varepsilon}{\delta^s} \rightharpoonup \frac{\varphi}{\delta^s} \tag{41}$$

in L^∞-weak-$\star$ as $\varepsilon \to 0$.

We can now use

$$\int_\Omega \underbrace{V_k u_{k,m}}_{L^1_{loc}} \underbrace{\eta_\varepsilon\varphi}_{L^\infty_c(\Omega)} \xrightarrow{k} \int_\Omega V u_m \eta_\varepsilon \varphi$$

Since $u_{k,m}/\delta^s \le \mathrm{G}(f_m)/\delta^s \in L^\infty(\Omega)$ we have

$$\int_\Omega \frac{u_{k,m}}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}(\varphi\eta_\varepsilon) \xrightarrow{k} \int_\Omega \frac{u_m}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}(\eta_\varepsilon\varphi) \tag{42}$$

Recovering the set of test functions

Since $u_{k,m}/\delta^s \le \mathrm{G}(f_m)/\delta^s \in L^\infty(\Omega)$ we have

$$\int_\Omega \frac{u_{k,m}}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}(\varphi\eta_\varepsilon) \to \int_\Omega \frac{u_m}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}\varphi \qquad (43)$$

Existence for $(f, V) \in L^\infty \times (L^1_{loc})_+$

Theorem

Let $f \in L^\infty(\Omega)$ and $0 \le V \in L^1_{loc}(\Omega)$. Then, there exists a unique $u \in L^1(\Omega)$ such that $Vu\delta^s \in L^1$ and

$$\int_\Omega u(-\Delta)^s_{\mathrm{RFL}}\varphi + \int_\Omega Vu\varphi = \int_\Omega f\varphi \qquad \forall \varphi \in \mathbb{X}^s.$$

Proof.

We have $u_k = \mathrm{G}_{V_k}(f)$ and using η_ε as a test function we get:

$$\int_\Omega \frac{u_k}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}(\eta_\varepsilon\varphi) + \int_\Omega V_k u_k \delta^s \frac{\eta_\varepsilon \varphi}{\delta^s} = \int_\Omega f\eta_\varepsilon\varphi$$

As $k \to +\infty$, we get $u_k \searrow u \le \mathrm{G}(f)$. Hence $u/\delta^s \le \mathrm{G}(f)/\delta^s \in L^\infty$. As above

$$\int_\Omega \frac{u}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}(\eta_\varepsilon\varphi) + \int_\Omega V u \delta^s \frac{\eta_\varepsilon \varphi}{\delta^s} = \int_\Omega f\eta_\varepsilon\varphi$$

Again, as $\varepsilon \to 0$

$$\int_\Omega \frac{u}{\delta^s}\delta^s(-\Delta)^s_{\mathrm{RFL}}\varphi + \int_\Omega V u \delta^s \frac{\eta_\varepsilon \varphi}{\delta^s} = \int_\Omega f\varphi.$$

□

Existence for $(f, V) \in L^1(\Omega, \delta^s) \times (L^1_{loc})_+$

Theorem

Let $f \in L^1(\Omega, \delta^s)$ and $0 \le V \in L^1_{loc}(\Omega)$. Then, there exists a unique solution of

$$\int_\Omega u(-\Delta)^s_{\mathrm{RFL}}\varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi \qquad \forall \varphi \in \mathbb{X}^s.$$

Proof.

Letting $u_m = \mathrm{G}_V(f_m)$ we have

$$\int_\Omega u_m(-\Delta)^s_{\mathrm{RFL}}\varphi + \int_\Omega V u_m \delta^s \frac{\varphi}{\delta^s} = \int_\Omega f_m \delta^s \frac{\varphi}{\delta^s}.$$

As

$$f_m \nearrow f \implies u_m \nearrow u \implies V u_m \delta^s \nearrow \underbrace{V u \delta^s}_{L^1} \text{ a.e. } \overset{\mathrm{MCT}}{\implies} V u_m \delta^s \nearrow V u \delta^s \text{ in } L^1$$

Thus

$$\int_\Omega u(-\Delta)^s_{\mathrm{RFL}}\varphi + \int_\Omega V u_m \delta^s \frac{\varphi}{\delta^s} = \int_\Omega f_m \delta^s \frac{\varphi}{\delta^s}.$$

□

A comment on boundary behaviour

When $V \geq C_V \delta^{-2+\varepsilon}$ then solutions are flatter than the usual estimate:

Theorem

Let $0 < \varepsilon < s$, $0 \leq f \in L^\infty$, $V(x) \geq C_V \delta(x)^{-2s} \geq 0$ *with* $C_V > -\gamma_{s+\varepsilon}$. *Then,*

$$\frac{u}{\delta^{s+\varepsilon}} \in L^\infty(\Omega). \tag{44}$$

This means that

$$\frac{u}{\delta^s}(x) \to 0, \qquad \text{as } x \to \partial\Omega. \tag{45}$$

Singular potential give flatter boundary behaviour.

Thank you for you attention.

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