Critical size homogenisation of a reaction-diffusion problems on perforated domains

maximal monotone graphs and general hole shapes

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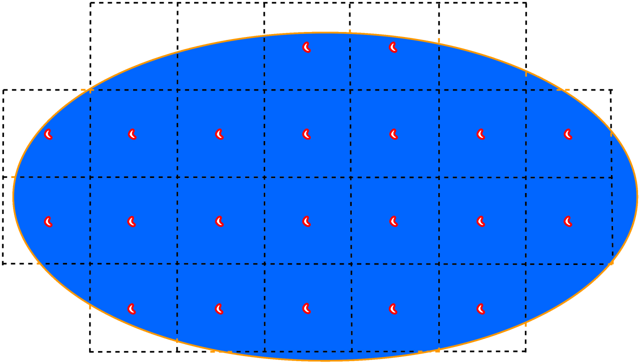
Aim

In this talk we study the limit problem

$$\begin{cases} -\Delta u_\varepsilon = f^\varepsilon & \Omega_\varepsilon, \\ \mathcal{B}_\varepsilon(u_\varepsilon) & S_\varepsilon, \\ u_\varepsilon = 0 & \partial\Omega, \end{cases} \tag{P$_\varepsilon$}$$

in the periodical geometry



Outline

For a generic particle $G_0 \subset Y = [-\frac{1}{2}, \frac{1}{2}]^n$, the geometry is given by

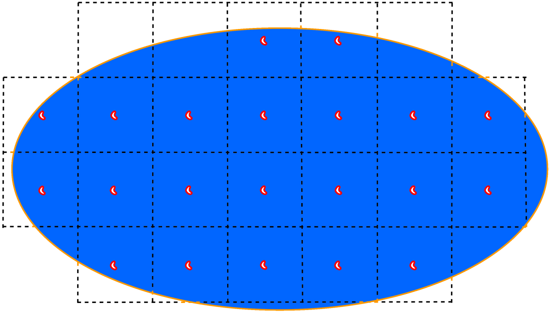


Figure: The domain Ω_ε.

$$G_i^\varepsilon = \varepsilon i + a_\varepsilon G_0, \qquad i \in \mathbb{Z}^n$$

$$\Upsilon_\varepsilon = \left\{ i \in \mathbb{Z}^n : \overline{G_i^\varepsilon} \subset \Omega \right\},$$

$$G^\varepsilon = \bigcup_{i \in \Upsilon_\varepsilon} G_i^\varepsilon,$$

$$\Omega^\varepsilon = \Omega \setminus \overline{G^\varepsilon}$$

$$S^\varepsilon = \bigcup_{i \in \Upsilon_\varepsilon} \partial G_i^\varepsilon.$$

The relation between ε (periodicity scale) and a_ε (particle size) will be key.

Remark

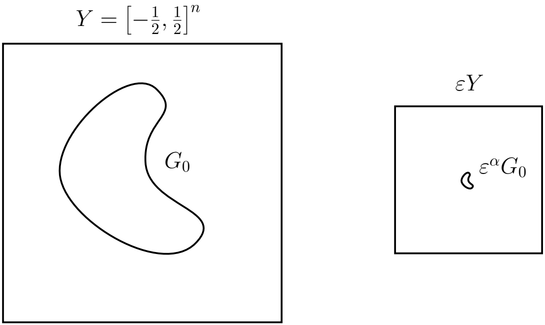
A similar problem where particles are on a $(n-1)$-dimensional manifold has also been studied. See, e.g., Gómez, Lobo, Pérez, and Shaposhnikova [GLPS11].

The particle scale factor a_ε

We will often consider

$$a_\varepsilon = C_0 \varepsilon^\alpha.$$

This parameter links the size with the repetition:



The coefficient α will be key.

Boundary condition $\mathcal{B}_\varepsilon(u_\varepsilon)$

We will cover several different boundary conditions $\mathcal{B}_\varepsilon(u_\varepsilon)$:

- Neumann-Robin: $\frac{\partial u_\varepsilon}{\partial n} + \beta(\varepsilon)\sigma(u_\varepsilon) = 0$
- Dirichlet: $u_\varepsilon = 0$
- Signorini:
  $(\frac{\partial u_\varepsilon}{\partial n} + \beta(\varepsilon)\sigma_0(u_\varepsilon))u_\varepsilon = 0, \quad \frac{\partial u_\varepsilon}{\partial n} + \beta(\varepsilon)\sigma_0(u_\varepsilon) \geq 0 \quad \text{and} \quad u_\varepsilon \geq 0.$

where σ non-decreasing and $\sigma(0) = 0$ (same for σ_0).
These three cases can be covered by the general formulation

$$-\frac{\partial u_\varepsilon}{\partial n} \in \beta(\varepsilon)\sigma(u_\varepsilon)$$

where σ is a maximal monotone graph.

The Dirichlet case:

$$\sigma(u) = \begin{cases} \emptyset & u \neq 0, \\ \mathbb{R} & u = 0. \end{cases}$$

The Signorini case:

$$\sigma(u) = \begin{cases} \emptyset & u < 0, \\ (-\infty, 0] & u = 0, \\ \sigma_0(u) & u > 0. \end{cases}$$

Weak formulation and bounds

For the Neumann problem we work on the functional setting We work on the domain

$$H^1(\Omega_\varepsilon, \partial\Omega) = \overline{\{u \in \mathcal{C}^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\}}^{H^1(\Omega_\varepsilon)}.$$

We define a weak solution of (P_ε) as:

$u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(u_\varepsilon) v = \int_{\Omega_\varepsilon} f^\varepsilon v \qquad (\mathrm{W}_\varepsilon)$$

for all $v \in H^1(\Omega_\varepsilon, \partial\Omega)$.

It is easy to see that, if σ is nondecreasing $\sigma(0) = 0$, then

$$\|\nabla u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \|f^\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

where C does not depend on ε.

Extension and limit

It is known that there exists a family of equicontinuous extension operators

$$P_\varepsilon : H^1(\Omega_\varepsilon, \partial\Omega) \to H^1_0(\Omega), \qquad P_\varepsilon(w)\Big|_{\Omega_\varepsilon} = w$$

By compactness $P_\varepsilon(u_\varepsilon) \rightharpoonup u$ in $H^1_0(\Omega)$.

The aim is to characterise u as the solution of a homogeneous problem (P).

Outline

Convergence of the averages

The homogenisation of the non-linear term for the subcritical cases is based on the fact that

Theorem

Let $a_\varepsilon = C_0 \varepsilon^\alpha$ *with* $1 \le \alpha < \frac{n}{n-p}$ *and* $W^{1,p}(\Omega) \ni v_\varepsilon \to v$ *in* $L^p(\Omega)$. *Then*

$$\frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} v_\varepsilon \, dH_{n-1} \longrightarrow \frac{1}{|\Omega|} \int_\Omega v \, dx$$

The coefficient $\beta(\varepsilon)$

Going back we want $\int_{S_\varepsilon}$ to behave in the limit of (W_ε), we need to have

$$\beta(\varepsilon) \int_{S_\varepsilon} \sim \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon}$$

First, we estimate the number of holes $|\Upsilon_\varepsilon|$.

$$\left| \Omega - \bigcup_{j \in \Upsilon_\varepsilon} (\varepsilon j + \varepsilon Y) \right| \to 0, \implies |\Upsilon_\varepsilon| \sim \varepsilon^{-n}.$$

Therefore

$$|S_\varepsilon| = \left| \bigcup_{j \in \Upsilon_\varepsilon} (j\varepsilon + a_\varepsilon \partial G_0) \right| = |\Upsilon_\varepsilon| a_\varepsilon^{n-1} |\partial G_0| \sim \varepsilon^{-n} a_\varepsilon^{n-1}$$

we define

$$\beta^*(\varepsilon) \sim \varepsilon^n a_\varepsilon^{1-n} \sim \varepsilon^{n-\alpha(n-1)}. \tag{1}$$

Often $\beta(\varepsilon) = \varepsilon^{-\gamma}$.
If $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$ reaction vanishes.
If $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ the reaction is dominating. The limit is $\sigma(u) = 0$.

The sub-critical cases: historial overview

This problem has been extensively studied in the 20th and 21th centuries.
Letting $n \geq 3$, $a_\varepsilon = C_0 \varepsilon^\alpha$ and $\beta(\varepsilon) = \varepsilon^{n-\alpha(n-1)}$:

- The case $a_\varepsilon \sim \varepsilon$ ($\alpha = 1$) was studied
  - In the Dirichlet case by Cioranescu and Saint Jean Paulin [CS79]:

    $$-\nabla \cdot (A^0 \nabla u) = f$$

    Matrix A^0 is usually called "effective diffusion" matrix.
  - In the Neumann case by Cioranescu and Donato [CD88a]:

    $$-\nabla \cdot (A^0 \nabla u) + C\sigma(u) = f$$

- The case $\varepsilon \gg a_\varepsilon \gg a_\varepsilon^\star = \varepsilon^{n/(n-2)}$ ($1 < \alpha < \frac{n}{n-2}$) was studied
  - In the Dirichlet case by Hruslov [Hru72]
  - For Neumann case by Conca and Donato [CD88b]:

    $$-\Delta u + C\sigma(u) = f$$

Super-critical case

The question is: does the same behaviour hold for $\alpha \geq n/(n-2)$ and we need a better technique?

No, it is relatively simple to study the case $a_\varepsilon = C_0 \varepsilon^\alpha \ll a_\varepsilon^\star = \varepsilon^{n/(n-2)}$ $(\alpha > \frac{n}{n-2})$ (see, e.g. Zubova and Shaposhnikova [ZS13])

the reaction term vanishes even for the correct scaling:

$$-\Delta u = f$$

Outline

The critical case: a “terme étrange venu d’ailleurs”

The surprising case is that of $a_\varepsilon = C_0 \varepsilon^{n/(n-2)}$:

- For the Neumann problem with $\sigma(u) = 0$ by Marchenko and Hruslov [MH74]
- For the Dirichlet problem by Cioranescu and Murat [CM82a; CM82b; CM97] who got
$$-\Delta u + \lambda u = f$$
this strange term comes out of the homogeneous Dirichlet condition.
- For σ smooth, $n = 3$ and G_0 a ball Goncharenko [Gon97] proved by Γ-convergence
$$-\Delta u + CH(u) = f$$
where H (different from $\sigma(u)$) is a solution of a functional equation of the form
$$C_n H(u) = \sigma(u - H(u)), \qquad \forall u \in \mathbb{R}.$$
Later studied in $n \geq 3$ by Oleinik, Shaposhnikova, Zubova, Podolskii, etc...
- The Signorini problem by Jäger, Neuss-Radu, and Shaposhnikova [JNS14]:
$$H(u) = \begin{cases} \lambda u & u \leq 0 \\ H_0(u) & u > 0 \end{cases}, \quad C_n H_0(u) = \sigma_0(u - H_0(u))$$

Goncharenko's strange term is smooth

Taking a derivative in the equation

$$C_n H(u) = \sigma(u - H(u)), \qquad \forall u \in \mathbb{R}$$

we obtain

$$C_n H'(u) = \sigma'(u - H(u))(1 - H'(u))$$

Hence

$$H'(u) = \frac{\sigma'(u - H(u))}{C_n + \sigma'(u - H(u))} \in [0, 1].$$

Therefore, H is non-decreasing and Lipschitz of constant at most 1.

This property is preserved also for the Signorini case.

The appearance of an "strange terms" satisfying similar equations is true for different critical-size homogenisation problems (book in preparation).

Since H is problem $-\Delta u + CH(u) = f$ is somewhat better in some contexts, e.g. Chemical Engineering (effectiveness).

Outline

Homogenisation with maximal monotone graphs

The main result of this section is that, for $1 < p < n$, $a_\varepsilon = C_0 \varepsilon^\alpha, \alpha = n/(n-p)$ and G_0 a ball, one can study the limit of solutions of problem

$$\begin{cases} -\Delta_p u_\varepsilon = f & \Omega_\varepsilon, \\ -|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot n \in \beta(\varepsilon) \sigma(u_\varepsilon) & S_\varepsilon, \\ u_\varepsilon = 0 & \partial\Omega, \end{cases} \tag{P_ε}$$

where $\beta(\varepsilon) = \varepsilon^{-\alpha(p-1)} = \varepsilon^{-\gamma}$. The limit is the solution of

$$\begin{cases} -\Delta_p u + A_0 |H(u)|^{p-2} H(u) = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{P}$$

where $H : \mathrm{Dom}(H) \subset \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous[2] and the unique solution of

$$B_0 |H(u)|^{p-2} H(u) \in \sigma(u - H(u)), \qquad \forall u \in \mathrm{Dom}(H). \tag{H}$$

This is the main result in [DGPS19]. The precise values of the constants are here

[2] For $p = 2$ it is Lipschitz of constant 1

Maximal monotone graphs and convexity

- A graph $\sigma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called monotone if

$$(\xi - \eta)(x - y) \geq 0, \qquad \forall x, y \in \mathbb{R}, \xi \in \sigma(x), \eta \in \sigma(y).$$

It is called maximal if there is no other monotone graph $\widetilde{\sigma}$ such that $\sigma(x) \subset \widetilde{\sigma}(x)$ for all $x \in \mathbb{R}$.

- Given a convex smooth function we have

$$\begin{aligned}\Psi(x) - \Psi(y) &= \Psi(x) - \left(\Psi(x) + \Psi'(x)(y-x) + \Psi''(\zeta)\frac{(y-x)^2}{2}\right) \\ &\leq \Psi'(x)(x-y).\end{aligned}$$

- Given a convex function $\Psi : \mathbb{R} \to (-\infty, +\infty]$, we define the subdifferential

$$\partial\Psi(x) = \{\xi \in \mathbb{R} : \Psi(x) - \Psi(y) \leq \xi(x-y) \, \forall y \in \mathbb{R}\}.$$

The subdifferential of a convex function is a maximal monotone operator.

- For every m.m.g. there exists a convex function Ψ such that $\sigma = \partial\Psi$.

Formally $\Psi(s) = \int_0^s \sigma(\tau) d\tau$.

Analysis of equation (H)

Let us go back to the equation

$$B_0 |H(u)|^{p-2} H(u) \in \sigma(u - H(u)), \qquad \forall u \in \mathrm{Dom}(H). \tag{H}$$

In the examples

The Dirichlet case:

$$\sigma(u) = \begin{cases} \emptyset & u \neq 0, \\ \mathbb{R} & u = 0. \end{cases}$$

In order to have a solution of (H) we need that $\sigma(u - H(u)) \neq \emptyset$.

Therefore $H(u) = u$.

We extend the celebrated result by Cioranescu and Murat [CM97]:

$$-\Delta_p u + A_0 |u|^{p-2} u = f.$$

The Signorini case:

$$\sigma(u) = \begin{cases} \emptyset & u < 0, \\ (-\infty, 0] & u = 0, \\ \sigma_0(u) & u > 0. \end{cases}$$

Similarly, we recover the result by [JNS14].

Idea of the proof

Definition of solution. Let us start by $\sigma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a maximal monotone graph. Then there exists $\Psi : \mathbb{R} \to (-\infty, +\infty]$ convex such that $\sigma = \partial \Psi$.

We define the energy functional over $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ as

$$J_\varepsilon(v) = \begin{cases} \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla v|^p + \beta(\varepsilon) \int_{S_\varepsilon} \Psi(v) - \int_{\Omega_\varepsilon} f^\varepsilon v, & \text{if } \Psi(v) \in L^1(S_\varepsilon) \\ +\infty & \text{otherwise.} \end{cases}$$

and the solution u_ε is defined as its unique minimizer.

Inequality formulation. It can be shown using monotonicity (see Brézis and Sibony [BS71] or Lions [Lio69]) that u_ε satisfies

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla (v - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) dS \geq \int_{\Omega_\varepsilon} f (v - u_\varepsilon) dx.$$

for all $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

Trading the boundary term. We extend Tartar's idea of the oscillating test functions (see Tartar [Tar10]).
Following the ideas in [OS96] we construct the auxiliary function

$$\begin{cases} \Delta_p w_\varepsilon^j = 0 & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 1 & x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0 & x \in \partial T_\varepsilon^j. \end{cases} \qquad W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \, j \in \Upsilon_\varepsilon; \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in \Upsilon_\varepsilon} T_\varepsilon^j, \end{cases}$$

and we use $v_\varepsilon = v - H(v) W_\varepsilon$ as a test function.

Since G_0 is a ball, and explicit formula can be obtained:

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}{(C_0 \varepsilon^\alpha)^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}, \qquad x \in T_\varepsilon^j, \setminus \overline{G_\varepsilon^j}, \tag{2}$$

Idea of the proof

Trading the critical-size boundary

Let $v_\varepsilon = v - hW_\varepsilon$. Then

$$\lim_{\varepsilon\to 0} \left(\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla (v_\varepsilon - u_\varepsilon) dx \right) = \lim_{\varepsilon \to 0} (I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon})$$

$$I_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla (v - u_\varepsilon) dx \qquad I_{3,\varepsilon} = -A_\varepsilon \varepsilon \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} |h|^{p-2} h (v - u_\varepsilon) dS,$$

$$I_{2,\varepsilon} = -\varepsilon^{-\gamma} \mathcal{B}_0 \int_{S_\varepsilon} |h|^{p-2} h (v - h - u_\varepsilon) dS$$

where A_ε is a known bounded sequence. Constants come from $\partial_n w_\varepsilon$.
Each term plays a role in the limit:

1. $I_{1,\varepsilon}$ gives the limit diffusion
2. $I_{2,\varepsilon}$ can be used, assuming (H), to cancel out the integral over S_ε in the weak formulation. We use convexity: $\Psi(b) - \Psi(a) \le \xi(b-a)$ for any $\xi \in \sigma(b)$.
3. $I_{3,\varepsilon}$ gives the "strange term".

Outline

General shapes

The case of general shape of hole G_0 was studied by
Díaz, GC, Shaposhnikova and Zubova [DGSZ17]
(for the linear case with space dependent coefficients see [DGSZ19]).

We now consider the auxiliary function

$$\begin{cases} -\Delta_y \widehat{w} = 0 & \text{if } y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_y} \widehat{w} - C_0 \sigma(u - \widehat{w}) = 0, & \text{if } y \in \partial G_0, \\ \widehat{w} \to 0 & \text{as } |y| \to \infty. \end{cases}$$

We define $H_{G_0} : \mathbb{R} \to \mathbb{R}$, for $u \in \mathbb{R}$, as the capacity-type term:

$$H_{G_0}(u) := \int_{\partial G_0} \partial_{\nu_y} \widehat{w}(y; G_0, u) ds_y = C_0 \int_{\partial G_0} \sigma(u - \widehat{w}(y; G_0, u)) ds_y.$$

Then

$$\begin{cases} -\Delta u_\varepsilon = f & \Omega_\varepsilon \\ \partial_n u_\varepsilon + \beta(\varepsilon) u_\varepsilon = 0 & S_\varepsilon \end{cases} \quad \xrightarrow{\varepsilon \to 0} \quad -\Delta u + C_0^{n-2} H_{G_0}(u) = f \text{ in } \Omega.$$

Idea of the proof

Similarly to the previous case, we can rescale $\widehat{w}$

$$\begin{cases} -\Delta \widehat{w}_\varepsilon^j = 0 & \mathbb{R}^n \setminus G_\varepsilon^j, \\ \partial_n \widehat{w}_\varepsilon^j - \varepsilon^{-\gamma} \sigma(u - \widehat{w}_\varepsilon^j) = 0 & \partial G_\varepsilon^j \\ \widehat{w}_\varepsilon^j \to 0 & |x| \to +\infty \end{cases}$$

with the property

$$- \lim_{\varepsilon \to 0} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\frac{\varepsilon}{4}}} \left(\partial_\nu \widehat{w}_\varepsilon^j (x; G_0, \phi(P_\varepsilon^j)) \right) h_\varepsilon(x) ds = C_0^{n-2} \int_\Omega H(\phi(x))\, h(x) \,\mathrm{d}x.$$

In order to trade the integral S_ε into balls of radius ε we use the auxiliary test functions:

$$\begin{cases} \Delta w_\varepsilon^j = 0 & \text{if } x \in T^j_{\frac{\varepsilon}{4}} \setminus \overline{G_\varepsilon^j}, \\ \partial_{\nu_x} w_\varepsilon^j - \varepsilon^{-\gamma} \sigma(u - w_\varepsilon^j) = 0 & \text{if } x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0 & \text{if } x \in \partial T^j_{\frac{\varepsilon}{4}}, \end{cases}$$

and with the property $w_\varepsilon - \widehat{w}_\varepsilon^j \to 0$ with adequate estimates.

Extensions and open problems

Extensions (in preparation)

- The parabolic problem can be studied in a similar way.
- Critical case with dynamical boundary condition $\partial_t u + \partial_n u + \sigma(u_\varepsilon) = 0$ on $\partial\Omega$. This yields a strange term H solving a nonlinear ODE.

Open problems

- General shapes for $p = 2$ and maximal monotone graphs.
- Case of general shapes and $p \neq 2$.
- Data $f \in L^1(\Omega)$
- Homogenisation of stochastic problem: either stochastic data, or stochastic domain

Thank you for you attention.

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Precise constants

The constants involved are

$$A_0 = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n, \qquad B_0 = \left(\frac{n-p}{C_0(p-1)}\right)^{p-1}$$

Go back