A short introduction to the fractional Laplacian

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- Relation to stochastic processes
- Fourier interpretation and Principal Value formula
- Elliptic equations: $(-\Delta)^s u + \mu u = f$ in $\mathbb{R}^d$
  - Weak solutions for $\mu > 0$
  - Weak solutions for $\mu = 0$
  - Basic properties of solutions
  - Viscosity solutions
- Alternative representations
  - The Caffarelli-Silvestre extension
  - The semi-group formula
- The fractional heat equation $u_t + (-\Delta)^s u = 0$ in $\mathbb{R}^d$
- Numerical analysis
- Bounded domains
  - Restricted fractional Laplacian
  - Censored fractional Laplacian
  - Spectral fractional Laplacian
  - A unified theory

Through this talk, we define the following operator with a *singular kernel*

$$(-\Delta)^s u(x) = \frac{C(d,s)}{2} \int_{\mathbb{R}^d} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{d+2s}} \, \mathrm{d}y. \tag{SK}$$

We will explain the name below. The popular choice is $C(d,s) = \left(\int_{\mathbb{R}^d} \frac{1-\cos\omega_1}{|\omega|^{d+2s}} \, \mathrm{d}\omega \right)^{-1}$. It will be justified below.

Let us think of an stochastic process discrete in space and time of the form

$$u(t+\tau, x) = \sum_{j \in \mathbb{Z}^n} u(t, x+hj) P(x+hj \to x).$$

τ is the time step, whereas h is the space step.

When $P(x \to x+hj) = \frac{1}{2d}$ for $j \in \{\pm e_1, \cdots, \pm e_d\}$ and 0 otherwise, we have Brownian motion and we recover the heat equation: $u_t - \Delta u = 0$

If we assume that there is long-range interaction

$$P(x+hj \to x) = \frac{c(d,s)}{|j|^{d+2s}}$$

we show now that we formally recover $u_t + (-\Delta)^s u = 0$

$$
\begin{aligned}
\frac{u(t+\tau,x)-u(t,x)}{\tau} &= \frac{c(d,s)}{\tau}\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x+hj)-u(t,x)}{|j|^{d+2s}}\\
&= \frac{c(d,s)}{2\tau}\left(\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x+hj)-u(t,x)}{|j|^{d+2s}}+\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x+hj)-u(t,x)}{|j|^{d+2s}}\right)\\
&= \frac{c(d,s)}{2\tau}\left(\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x+hj)-u(t,x)}{|j|^{d+2s}}+\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x-hj)-u(t,x)}{|-j|^{d+2s}}\right)\\
&= \frac{c(d,s)}{2\tau}\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x+hj)+u(t,x-hj)-2u(t,x)}{|j|^{d+2s}}\\
&= \frac{h^{2s}}{\tau}\frac{c(d,s)}{2}\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\frac{u(t,x+hj)+u(t,x-hj)-2u(t,x)}{|hj|^{d+2s}}h^d\\
&\sim \frac{h^{2s}}{\tau}\frac{c(d,s)}{2}\int_{\mathbb{R}^d}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{d+2s}}\,\mathrm{d}y
\end{aligned}
$$

Taking $\tau = h^{2s}$ and letting $h \to 0$ we recover

$$u_t = -(-\Delta)^s u.$$

Definition (Lévy process)

A stochastic process $X = \{X_t : t \geq 0\}$ is called a Lévy process if

1. $X_0 = 0$ almost surely
2. **Independence of increments:** For any $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n_1}}$ are independent.
3. **Stationary increments:** For any $t_1 < t_2$, $X_{t_2} - X_{t_1}$ is equal in distribution to $X_{t_2 - t_1}$
4. **Continuity in probability:** For any $\varepsilon > 0$ and $t \geq 0$, $\lim_{h \to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$.

If X_t is a Lévy process, then $p_t(x, B) = P(X_t + x \in B)$ is a Markov transition function (see, e.g. [Schilling 2016, Lemma 4.4]).
In particular, the $2s$-stable process (sometimes called Lévy-flights) is given by the transition function

$$p(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi - t|\xi|^{2s}} \, \mathrm{d}\xi. \tag{P}$$

These were introduced in [Blumenthal, Getoor, and Ray 1961].
By the Chapman-Kolmogorov formula $p_t(x, B) = \int_B p(t, y - x) \, \mathrm{d}y$.
The FL is the infinitesimal generator of the process (see, e.g., [Kühn and Schilling 2019]):

$$(-\Delta)^s u(x_0) = \lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}[u(x_0 + X_t)] - u(x_0) \right)$$

The corresponding Feynman-Kac using fractional random process exist.

Fractional spaces and operators have recently be used in many applications.

[Di Nezza, Palatucci, and Valdinoci 2012] list:

- thin obstacle problem [85,68]
- optimization [37], finance [26]
- phase transitions [2,14,86,40,45]
- stratified materials [81,23,24]
- anomalous diffusion [67,96,64]
- crystal dislocation [90,47,8],
- soft thin films [56],
- semipermeable membranes and flame propagation [15]
- conservation laws [9],
- ultra-relativistic limits of quantum mechanics [41],
- quasi-geostrophic flows [63,27,21],
- multiple scattering [36,25,49],
- minimal surfaces [16,20],
- materials science [4],
- water waves [79,98,97,32,29,72,33,-34,31,30,42,50,73,35],
- elliptic problems with measure data [70,53],
- non-uniformly elliptic problems [39],
- gradient potential theory [71]
- singular set of minima of variational functionals [69,55].

Lemma 1

Let $s > 0$ and $u \in C^{2s+\varepsilon}(\mathbb{R}^d)$. Then, $(-\Delta)^s u(x)$ is well defined.

Now we prove this representation is uniform integrable, by splitting the integral. The integral is easy for y large

$$\int_{\mathbb{R}^d \setminus B_1(0)} \left| \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{d+2s}} \right| \mathrm{d}y \leq 3\|u\|_{L^\infty} \int_{\mathbb{R}^d \setminus B_1(0)} \frac{1}{|y|^{d+2s}} \mathrm{d}y$$

Close to zero we take a Taylor expansion

$$\int_{B_1(0)} \left| \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{d+2s}} \right| \mathrm{d}y \leq C[u]_{C^{2s+\varepsilon}} \int_{B_1(0) \setminus B_r(0)} \frac{|x-y|^{2s+\varepsilon}}{|x-y|^{d+2s}} \mathrm{d}y$$

Theorem 2 Proposition 2.1.7 in [Silvestre 2005]

We have $(-\Delta)^s : C^{2s+\varepsilon}(\mathbb{R}^d) \to C^{\varepsilon}(\mathbb{R}^d)$.

We can replace this with the Fourier transform $\mathcal{F}$. We recall that

$$\mathcal{F}[-\Delta u](\xi) = -\int_{\mathbb{R}^d} (\Delta u)(x) e^{-2\pi i x\cdot\xi}\,\mathrm{d}x = -\int_{\mathbb{R}^d} u(x)\Delta_x e^{-2\pi i x\cdot\xi}\,\mathrm{d}x = (2\pi|\xi|)^2\mathcal{F}[u].$$

Theorem 3

$$\mathcal{F}[(-\Delta)^s u] = (2\pi|\xi|)^{2s}\mathcal{F}[u]. \tag{F}$$

$$\begin{aligned}\mathcal{F}[(-\Delta)^s u] &= \frac{C(d,s)}{2}\int_{\mathbb{R}^d} \frac{2\mathcal{F}[u] - \mathcal{F}[u(\cdot+y)] - \mathcal{F}[u(\cdot-y)]}{|y|^{d+2s}}\,\mathrm{d}y \\ &= \left(\frac{C(d,s)}{2}\int_{\mathbb{R}^d}\frac{2 - e^{2\pi i\xi\cdot y} - e^{-2\pi i \xi\cdot y}}{|y|^{d+2s}}\,\mathrm{d}y\right)\widehat{u}(\xi)\end{aligned}$$

Notice that

$$\frac{1}{2}\int_{\mathbb{R}^d}\frac{2 - e^{2\pi i\xi\cdot y} - e^{-2\pi i\xi\cdot y}}{|y|^{d+2s}}\,\mathrm{d}y = \int_{\mathbb{R}^d}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{d+2s}}\,\mathrm{d}y = (2\pi|\xi|)^{2s}\int_{\mathbb{R}^d}\frac{1-\cos\omega_1}{|\omega|^{d+2s}}\,\mathrm{d}\omega.$$

This drives the choice of $C(d,s)$.

A Principal Value formula

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Coming from the Fourier transform it was shown in [Landkof 1972, p.45] or [Stein 1970]) that

Theorem 4

$$(-\Delta)^s u(x) = C(d,s) \lim_{r\to 0+} \int_{\mathbb{R}^d \setminus B_r} \frac{u(x)-u(x+y)}{|y|^{d+2s}}\,\mathrm{d}y. \tag{PV}$$

Proof. With our definition

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus B_r(0)} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{d+2s}}\,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^d \setminus B_r(0)} \frac{u(x)-u(x+y)}{|y|^{d+2s}}\,\mathrm{d}y + \int_{\mathbb{R}^d \setminus B_r(0)} \frac{u(x)-u(x-y)}{|y|^{d+2s}}\,\mathrm{d}y \\
&\quad = 2\int_{\mathbb{R}^d \setminus B_r(0)} \frac{u(x)-u(x+y)}{|y|^{d+2s}}\,\mathrm{d}y. \quad \square
\end{aligned}$$

Some authors write

$$(-\Delta)^s u(x) = C(d,s)\,\mathrm{P.V.} \int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2s}}\,\mathrm{d}y.$$

Recall for the usual Laplacian $\int_{\mathbb{R}^d} \varphi(-\Delta u) = \int_{\mathbb{R}^d} \nabla\varphi\nabla u$.

Theorem 5

For $u, \varphi \in C_c^\infty(\mathbb{R}^d)$ we have that

$$\int_{\mathbb{R}^d} \varphi(-\Delta)^s u \,\mathrm{d}x = \underbrace{\frac{C(n,s)}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\varphi(x)-\varphi(y))(u(x)-u(y))}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x}_{\mathcal{E}_s(u,\varphi)} . \tag{W}$$

People in probably call $\mathcal{E}_s$ the Dirichlet form of X_t (see [Blumenthal and Getoor 1968]).
Proof. We can integrate by parts (noticing that we are removing a strip around $x = y$)

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d\setminus B_r(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_r(x)} \varphi(x)\frac{u(x)-u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_r(x)} \varphi(y)\frac{u(y)-u(x)}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_r(x)} \frac{(\varphi(x)-\varphi(y))(u(x)-u(y))}{|x-y|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}x \qquad \square
\end{aligned}$$

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Recall for the usual Laplacian, the weak formulation of $-\Delta u + \mu u = f$ is

$$u \in H^1(\mathbb{R}^d) \text{ such that } \qquad \underbrace{\underbrace{\int_{\mathbb{R}^d} \nabla u \nabla \varphi}_{\mathcal{E}_1(u,\varphi)} + \mu \underbrace{\int_{\mathbb{R}^d} u\varphi}_{(u,\varphi)_{L^2(\mathbb{R}^d)}}}_{a_\mu(u,v)} = \int_{\mathbb{R}^d} f\varphi \qquad \forall \varphi \in H^1(\mathbb{R}^d).$$

For $\mu > 0$, $a_\mu(u,\varphi)$ is bilinear, coercive and continuous in $H^1(\mathbb{R}^d)$,
so there exists a unique solution in $H^1(\mathbb{R}^d)$ by Lax-Milgram.

For $\mu = 0$, $E_1(u,\varphi)$ is not coercive in H^1. We cannot apply Lax-Milgram in H^1.
There exists a solution with $\nabla u \in L^2$, but $u \notin L^2$.

As for H^1, we define

$$[u]_{H^s(\mathbb{R}^d)} = \sqrt{\mathcal{E}_s(u,u)} = \left(\frac{C(d,s)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{2}},$$

We define $\dot{H}^s(\mathbb{R}^d) = \left\{ u \text{ measurable} : [u]_{W^{s,p}(\mathbb{R}^d)} < \infty \right\}$
These sets are not nice: [Brasco, Gómez-Castro, and Vázquez 2021].

Theorem 6

$\mathcal{E}_s : \dot{H}^s \times \dot{H}^s \to \mathbb{R}$ is well-defined, bilinear, symmetric and $|\mathcal{E}_s(u,\varphi)| \le [u]_{H^s}[\varphi]_{H^s}$.

It is easy to see that $(-\Delta)^s$ is the subdifferential of $J_s(u) = [u]^2_{H^s(\mathbb{R}^d)}$, i.e.
$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} J[u+\varepsilon\varphi] = \mathcal{E}_s(u,\varphi)$.

Lastly, we define $H^s(\mathbb{R}^d) = L^2(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ with $\|u\|_{H^s} = \|u\|_{L^2} + [u]_{H^s}$.

Theorem 7

$H^s(\mathbb{R}^d)$ is a Hilbert space. $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is a dense subset.

Due to (W)

$$\begin{cases} (-\Delta)^s u + \mu u = f \in L^2(\mathbb{R}^d), \\ u \in H^s(\mathbb{R}^d) \end{cases} \iff \underbrace{\mathcal{E}_s(u,\varphi) + \mu \int_{\mathbb{R}^d} u\varphi}_{a_{s,\mu}(u,\varphi)} = \int_{\mathbb{R}^d} f\varphi, \quad \forall \varphi \in H^s(\mathbb{R}^d).$$

The weak formulation is

$$u \in H^s(\mathbb{R}^d) \text{ such that } \qquad a_{s,\mu}(u,\varphi) = \int_{\mathbb{R}^d} f\varphi, \qquad \forall \varphi \in H^s(\mathbb{R}^d). \qquad (\star)$$

For $\mu > 0$, $a_{s,\mu}$ is coercive in $H^s(\mathbb{R}^d)$. We can apply the Lax-Milgram theorem

Theorem 8

For $\mu > 0$ and $f \in L^2(\mathbb{R}^d)$, there exists a unique u such that $(\star)$.

Notice that the condition $u \in L^2(\mathbb{R}^d)$ introduces a boundary condition $u \to 0$ as $|x| \to 0$.

To solve $(-\Delta)^s u = f$. We can use the Fourier transform $\mathcal{F}$.
Case $d > 2s$. (i.e. $d = 2$ and $s < 1$ or $d \geq 3$), we have
$\mathcal{F}^{-1}[(2\pi|\xi|)^{-2s}](x) = D(d,s)|x|^{-d+2s}$. Hence

$$(-\Delta)^{-s} f = \mathcal{F}^{-1}\left[(2\pi|\xi|)^{-2s}\mathcal{F}[f]\right] = \mathcal{F}^{-1}[(2\pi|\xi|)^{-2s}] * f$$
$$= D(d,s)\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2s}}\,\mathrm{d}y.$$

This is called Riesz potential. Some authors denote $I_{2s}[f] = (-\Delta)^{-s}[f]$.
When it is defined, we have the solution $u = I_{2s}[f]$.

Notice that $(-\Delta)^s 1 = 0$. So $u(x) = c + I_{2s}[f]$ are solutions.

This approach can also be use to get a formula when $\mu > 0$.
However, $\mathcal{F}^{-1}[\frac{1}{(2\pi|\xi|)^{2s}+\mu}]$ does not have such a nice formula.

Case $d < 2s$. We have the same problem as for $-\Delta$ in dimensions $d = 1, 2$.
The Green function still exists, but it has different properties.

If $f \in L_c^\infty(\mathbb{R}^d)$, then $I_{2s}[f]$ is point-wise well-defined, and $I_{2s}[f] \to 0$ as $|x| \to \infty$.

The integrability of the Riesz potential is well-understood
(see, e.g., [Schikorra, Spector, and Van Schaftingen 2017])

Theorem 9 (Sobolev)

If $2s < d$ and $1 \leq p < \frac{d}{2s}$, then $I_{2s}(f)$ absolutely converges.

Furthermore, if $p > 1$ then $I_{2s} : L^p(\mathbb{R}^d) \to L^{\frac{dp}{d-2sp}}$.

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Using basically the properties of the Riesz operator

Theorem 10 Propositions 2.8 and 2.9 in [Silvestre 2007]

If $(-\Delta)^s w = h$ in $\mathbb{R}^d$. For $\beta \in (0, 2s)$

$$\|w\|_{C^\beta(\mathbb{R}^d)} \leq C(\|w\|_{L^\infty(\mathbb{R}^d)} + \|h\|_{L^\infty(\mathbb{R}^d)})$$

If $\beta, \beta + 2s \notin \mathbb{Z}$ then

$$\|w\|_{C^{\beta+2s}(\mathbb{R}^d)} \leq C(\|w\|_{C^\beta(\mathbb{R}^d)} + \|h\|_{C^\beta(\mathbb{R}^d)})$$

With a similar argument

Theorem 11 [Ros-Oton and Serra 2014]

If $(-\Delta)^s w = h$ in B_1. For $\beta \in (0, 2s)$

$$\|w\|_{C^\beta(\overline{B}_{1/2})} \leq C(\|w\|_{L^\infty(\mathbb{R}^d)} + \|h\|_{L^\infty(B_1)})$$

If $\beta, \beta + 2s \notin \mathbb{Z}$ then

$$\|w\|_{C^{\beta+2s}(\overline{B}_{1/2})} \leq C(\|w\|_{C^\beta(\mathbb{R}^d)} + \|h\|_{C^\beta(\overline{B}_1)})$$

Weak maximum principle for $\mu = 0$

Theorem 12 (Weak maximum principle)

Let $u \in H^s(\mathbb{R}^d)$ be such that $(-\Delta)^s u \leq 0$ is the sense that $\mathcal{E}_s(u, \varphi) \leq 0$ for all $\varphi \geq 0$.
Then $u \leq 0$ a.e. in $\mathbb{R}^d$.

Due to (W) , we know that

$$\int_{\mathbb{R}^d} (-\Delta)^s u \varphi \, \mathrm{d}x = \int_{\mathbb{R}^d} u (-\Delta)^s \varphi \, \mathrm{d}x, \qquad \forall u, \varphi \in C_c^\infty(\mathbb{R}^d). \tag{SA}$$

Proof. We deduce that

$$\int_{\mathbb{R}^d} u(-\Delta)^s \varphi = \mathcal{E}_s(u, \varphi) \leq 0.$$

Take $K \Subset \mathbb{R}^d$ and $\varphi = I_{2s}[\chi_K] \geq 0$, then $\int_K u \leq 0$. We deduce $u \leq 0$. □

Under some condition on u, we can define $(-\Delta)^s u \in D'(\mathbb{R}^d)$ as

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^d} u(x)(-\Delta)^s \varphi(x), \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d).$$

This condition is

$$\int_{\mathbb{R}^d} \frac{|u(x)|}{1+|x|^{d+2s}} \, \mathrm{d}x < \infty.$$

Theorem 13 (Weyl's lemma for the FL) [Fall 2016]

If $(-\Delta)^s u = 0$ in $\mathcal{D}'(\mathbb{R}^d)$. Then u is affine. If $s \in (0, \frac{1}{2}]$, then u is constant.

Written in functional terms

$$u \in \ker_{\mathcal{D}'(\mathbb{R}^d)} (-\Delta)^s \implies u(x) = \begin{cases} c_1 + v \cdot x & s \in (\frac{1}{2}, 1] \\ c_1 & s \in (0, \frac{1}{2}]. \end{cases}$$

Hence, for $d > 2s$ and if $u = I_{2s}[f]$ is a distributional solution of $(-\Delta)^s u = f$ the set of distributional solutions is $u = I_{2s}[f] + u_0$, where $u_0 \in \ker(-\Delta)^s$.

At a *global* maximum we have that $u(x) > u(x+y)$ hence

$$(-\Delta)^s u(x) = \frac{C(n+2s)}{2} \int_{\mathbb{R}^d} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{d+2s}} \, \mathrm{d}y \geq 0.$$

Unless u is constant, there exists a ball B where $u(x) > u(z)$ for $z \in B$.
Thus $(-\Delta)^s u(x) > 0$.

Theorem 14 (Strong maximum principle)

Let $u \in C^{2s+\varepsilon}(\mathbb{R}^d)$ be such that $(-\Delta)^s u \leq 0$.
Then, either u is constant or for each $x \in \mathbb{R}^d$, $u(x) < \sup_{\mathbb{R}^d} u$.

Integrating by parts twice, we deduce that

$$\int_{\mathbb{R}^d} u(-\Delta)^s \varphi = \int_{\mathbb{R}^d} f\varphi \qquad \forall \varphi \in X$$

In order to integrate by parts we still need to assume "$u \to 0$".
To avoid this space X, we can invert to

$$\int_{\mathbb{R}^d} u\psi = \int_{\mathbb{R}^d} f I_{2s}[\psi] \qquad \forall \psi \in D(I_{2s}) \cap L^\infty_c(\mathbb{R}^d) \tag{WD}$$

This is known as *weak-dual formulation*.
Taking $\psi = \mathrm{sign}_+(u)\chi_K$ we deduce

$$\int_K u_+ = \int_{\mathbb{R}^d} f I_{2s}[\mathrm{sign}_+(u)\chi_K] \le \int_{\mathbb{R}^d} f_+ I_{2s}[\chi_K] \le \|I_{2s}(\chi_K)\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} f_+.$$

This is a L^1_{loc} estimate and a comparison principle (weak maximum principle).

If $d > 2s$, $f \ge 0$ and $u = I_{2s}[f] \in L^1_{loc}$, then u is an solution of (WD) taking

$$u_k = I_{2s}[\min\{f, k\}\chi_{B_k}] \nearrow u$$

When f changes sign $u = I_{2s}[f_+] - I_{2s}[f_-]$ is a solution by linearity.

Theorem 15 (Kato inequality) [Chen and Véron 2014]

In the distributional sense

$$(-\Delta)^s |u| \leq \operatorname{sign}(u) (-\Delta)^s u.$$

This can be proved through a different inequality

Theorem 16 (Córdoba-Córdoba inequality) [Cordoba and Cordoba 2003]

Let $\varphi \in C^2(\mathbb{R})$ convex, such that $u, \varphi(u)$ are such that $(-\Delta)^s u, (-\Delta)^s \varphi(u)$ exist. Then

$$(-\Delta)^s \varphi(u) \leq \varphi'(u) (-\Delta)^s u$$

In [Caffarelli and Sire 2017], the authors extend these results to more general setting

The problem

$$(-\Delta)^s u + g(u) = f$$

with g non-decreasing and with adequate properties, can be treated by standard arguments. For example by minimising the energy

$$\mathcal{E}(u,u) + \int_{\mathbb{R}^d} \int_0^{u(x)} g(s)\,\mathrm{d}s\,\mathrm{d}x - \int_{\mathbb{R}^d} f(x)u(x)\,\mathrm{d}x.$$

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Consider the problem $-\Delta u + H(x, u, \nabla u) = f$.
We aim to replace weak derivatives by the derivatives of tangent functions.

Take $x_0 \in \mathbb{R}^d$, and u a classical subsolution, i.e. $-\Delta u + H(x, u, \nabla u) \leq f$.
Say φ is smooth and *touches* u *from above at* x_0 i.e. it is defined in a neighbourhood of U and

$$\varphi - u \geq 0, \qquad \varphi(x_0) - u(x_0) = 0.$$

Then x_0 is a minimum of $\varphi - u$, so

$$\nabla\varphi(x_0) - \nabla u(x_0) = 0, \qquad -(\Delta\varphi(x_0) - \Delta u(x_0)) \leq 0.$$

Then

$$-\Delta\varphi(x_0) + H(x_0, \varphi(x_0), \nabla\varphi(x_0)) \leq -\Delta u(x_0) + H(x_0, u_0, \nabla u(x_0)) \leq f(x_0).$$

Definition 17 (*viscosity sub-solution* of $-\Delta u + +H(x_0, \varphi(x_0), \nabla\varphi(x_0)) = 0$ in Ω)

Let $u \in C(\Omega)$ such that for any $x_0 \in \Omega$ and φ of class C^2 touching from above at x_0, we have

$$-\Delta\varphi(x_0) + H(x_0, \varphi(x_0), \nabla\varphi(x_0)) \leq f(x_0)$$

Similarly, there exists a notion of *viscosity super-solution,* with function φ touching from below.
A function is a *viscosity solution* if it is both a super- and sub-solution.
The definition of sub-solution can be extended to upper semi-continuous functions.
This notion was introduced in [Crandall and Lions 1983].

For the a general fully non-linear equation (think Hamilton-Jacobi)

$$F(x, u(x), Du(x), D^2u(x)) = 0$$

The example above is $F(x, s, p, X) = -\mathrm{trace}(X) + s - f(x)$.

To define viscosity sub/super-solutions we must assume that,
when $X - Y$ is positive definite, then $F(X) \leq F(Y)$.

There theory of viscosity solutions is well-understood [Crandall, Ishii, and Lions 1992].
Some key features:

1. **Stability.** if u_ε are solutions corresponding for F_ε and $u_\varepsilon \to u$ and $F_\varepsilon \to F$ uniformly, then u is a solution for F.
2. **Existence.** Typically achieved by taking $F_\varepsilon = F - \varepsilon\mathrm{trace}(X)$ (i.e, adding $-\varepsilon\Delta u_\varepsilon$). This method is known as *vanishing viscosity*, and it names the theory.
3. **Uniqueness.** This type of solutions (+ suitable boundary conditions) typically have a comparison principle.
4. **Regularity.** The standard theories are developed: [Caffarelli and Cabré 1995].

They are usually equivalent to weak solutions, when these can be defined.
For linear problems [Ishii 1995]. For the p-Laplace, e.g. [Medina and Ochoa 2019].

A nice introductory text to viscosity solutions is [Katzourakis 2015].

There is an equivalent notion of viscosity solution, introduced in [Korvenpää, Kuusi, and Lindgren 2019].

Definition 18 (viscosity super-solution of $(-\Delta)^s u = f$ in Ω)

$u : \mathbb{R}^d \to [-\infty, \infty]$ such that

1. $u < \infty$ a.e. in $\mathbb{R}^d$ and $u > -\infty$ a.e. in Ω
2. u is lower semi-continuous
3. For any $B_r(x_0) \subset \Omega$ and $\phi \in C^2(B_r(x_0))$ such that $\phi \leq u$ we have
$$(-\Delta)^s \varphi_r(x_0) \geq f(x_0)$$
where
$$\phi_r(x_0) = \begin{cases} \phi(x) & x \in B_r(x_0), \\ u(x) & x \in \mathbb{R}^d \setminus B_r(x_0). \end{cases}$$
4. $u_- = \max\{-u, 0\}$ is such that
$$u_- \in L^1_{loc}(\mathbb{R}^d) \qquad \int_{\mathbb{R}^d} \frac{u_-(x)}{(1+|x|)^{d+2s}}\, \mathrm{d}x < \infty$$

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Theorem 19 [Caffarelli and Silvestre 2007]

Let $u : \mathbb{R}^d \to \mathbb{R}$ and $U : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be the solution of

$$\begin{cases} -\mathrm{div}_{(x,y)}(y^{1-2s}\nabla_{(x,y)}U) = 0 & (x,y) \in \mathbb{R}^d \times \mathbb{R}, \\ U(x,0) = u(x) & x \in \mathbb{R}^d. \end{cases}$$

Then

$$(-\Delta)^s u(x) = C_E(d,s) \lim_{y \to 0+} \frac{U(x,y) - U(x,0)}{|y|^{2s}} \tag{CSE}$$

The proof can be done by Fourier transform, or via the representation via Poisson kernel

$$U(x,y) = \int_{\mathbb{R}^d} P(x-\xi, y)u(\xi)\,\mathrm{d}\xi, \qquad P(x,y) = \widetilde{C}_E \frac{y^{1-2s}}{(|x|^2 + |y|^2)^{\frac{d-2s}{2}}}$$

and making the computation directly.

Corollary 20 (Harnack inequality) [Caffarelli and Silvestre 2007]

Assume $u \geq 0$ and that $(-\Delta)^s u = 0$ in B_r. Then

$$\sup_{B_{\frac{r}{2}}} u \leq C \inf_{B_{\frac{r}{2}}} u.$$

The idea is to use the Harnack inequality for U in
[Caffarelli, Kocan, Crandall, and Święch 1996].

A proof of this result through stochastic processes is older: [Bass and Kassmann 2004].

A Harnack-type inequality is the first step towards $u \in C^\alpha$.
[DiBenedetto 1983; Caffarelli and Cabré 1995; Urbano 2008].

In [Cabré and Sire 2014a; Cabré and Sire 2014b] the authors use the CSE to study the regularity of

$$(-\Delta)^s u = f(u).$$

They deal with $d = 1$ or $d > 1$ and radial solutions.

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The semi-group formula: $(-\Delta)^s$ through $e^{t\Delta}$

Let $e^{t\Delta}$ be the heat semigroup,
i.e. $u(t) = e^{t\Delta}[u_0]$ as the solution of $u_t = \Delta u$ with $u(0) = u_0$.

Theorem 21 [Bochner 1949]

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}[u](x) - u(x) \right) \frac{\mathrm{d}t}{t^{1+s}}. \qquad \text{(SG)}$$

The advantage is that the Riesz potential can also be similarly approximated

$$I_{2s}[f] = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta}[u](x) \frac{\mathrm{d}t}{t^{1-s}}.$$

Since $e^{t\Delta}$ is well understood. This formula is very powerful to show estimates and regularisation properties. A nice survey is [Stinga 2019].

The fractional (s,p)-Laplacian

The non-local equivalent of the p-Laplacian $-\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is

$$(-\Delta)^s_p u(x) = C_1 \, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{\Phi_p(u(x)-u(y))}{|x-y|^{d+sp}} \, \mathrm{d}y, \qquad \Phi_p(t) = |t|^{p-2}t$$

where $p \in (1,\infty)$ and $s \in (0,1)$, and $C_1 = C_1(n,s,p) > 0$.
It was introduced in [Chambolle, Lindgren, and Monneau 2012],
although it belongs to an older class previously studied.
Some properties:

1. Is related to $W^{s,p}$ instead for H^s
2. Viscosity solutions can be found in
   [Ishii and Nakamura 2010; Bjorland, Caffarelli, and Figalli 2012],
   [Korvenpää, Kuusi, and Lindgren 2019].
3. For the evolution see [Mazón, Rossi, and Toledo 2016; Vázquez 2016; Vázquez 2020]
4. Admits some representations similar to (SG) and (CSE). See [Teso, GC, and Vázquez 2020].

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Let us take a look at $u_t + (-\Delta)^s u = 0$ with $u(0) = u_0$. We have two options:

1. Existence of a heat kernel through coming from the stochastic process
2. Fourier transform approach
3. Semi-group approach

A detailed theory of well-posedness and regularity can be found in [Bonforte, Sire, and Vázquez 2017]
This works contain a detailed literature review.

Taking the Fourier transform is space $\widehat{u}(t,\xi) = \mathcal{F}[u(t,\cdot)](\xi)$ we rewrite the equation

$$\frac{\partial \widehat{u}}{\partial t} = -(2\pi|\xi|)^{2s}\widehat{u}(\xi).$$

We have an ODE in each *mode*. Hence

$$\widehat{u}(t,\xi) = e^{-t(2\pi|\xi|)^{2s}}\widehat{u}(0,\xi).$$

Finally,

$$u(t,x) = \mathcal{F}^{-1}\big[e^{-t(2\pi|\xi|)^{2s}}\big] * u_0$$

Going back to stochastic process, notice the relation to (P).

The key result for the semi-group theory (in the linear setting) is

Theorem 22 [Hille 1952]

If X is a Banach space and A is an operator such that

1. $A : D(A) \subset X \to X$ is linear, closed and $D(A)$ is dense
2. $(I + \lambda A)^{-1} : X \to X$ and
3. The operator is accretive: $\|(I + \lambda A)^{-1} u\|_X \le \|u\|_X$ for all $\lambda > 0$ and $u \in X$.

Then, for $u_0 \in D(A)$,

$$u(t) = \lim_{n \to \infty} (I + \tfrac{t}{n} A)^{-n} u_0$$

is defined for all $t \ge 0$, and it is a solution of $\frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0$ and $u(0) = u_0$.

Notice this is the implicit time discretisation

$$\frac{u_n - u_{n-1}}{h} + Au_n = 0$$

which we can rewrite $(I + hA)u_n = u_{n-1}$.

[Crandall and Liggett 1971] proved the analogous for non-linear operators and Banach spaces.

Proof. When we take $X = L^2(\mathbb{R}^d)$ and $u \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\|u + \lambda(-\Delta)^s u\|_{L^2}^2 &= \int_{\mathbb{R}^d} (u + \lambda(-\Delta)^s u)^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u^2 + 2\lambda \int_{\mathbb{R}^d} u(-\Delta)^s u \,\mathrm{d}x + \lambda^2 \int_{\mathbb{R}^d} |(-\Delta)^s u|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} u^2 + 2\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}y \,\mathrm{d}x + \lambda^2 \int_{\mathbb{R}^d} |(-\Delta)^s u|^2 \,\mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} u^2.
\end{aligned}$$

Letting $f = u + \lambda(-\Delta)^s u$ we have

$$\|f\|_{L^2(\mathbb{R}^d)} \geq \|(I + \lambda(-\Delta)^s)^{-1} f\|_{L^2(\mathbb{R}^d)}.$$

By density, this holds for all $f \in L^2(\mathbb{R}^d)$. □

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Finite differences.
The methods look like

$$(-\Delta)^s_h u(x) = \sum_{j\in\mathbb{Z}^n} \omega_j (u(x) - u(x+hj)).$$

- The discretisation of the different representation gives different weights. Quadrature formulas in (PV), (CSE), and (SG).
- If $\omega_j \geq 0$ for all j, then the method is stable and has nice properties.
- Finite difference solutions usually converge to the viscosity solution.
- See [Huang and Oberman 2014; Cusimano, Teso, and Gerardo-Giorda 2020]

Finite elements

- We take a triangulation, the corresponding finite elements V_h, and restrict $\mathcal{E}_s$ to $V_h \times V_h$.
- Convergence typically occurs in $H^s(\mathbb{R}^d)$.
- Finite elements solutions typically converge to weak solutions.

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If our stochastic process X_t must remain in a bounded set Ω,
we need to deal with jumps across $\partial\Omega$.

A nice stochastic discussion can be found in [Garbaczewski and Stephanovich 2019].
Broadly speaking

1. We can allow particles to jump anywhere,
but they are frozen when they reach some point $x \in \mathbb{R}^d \setminus \Omega$.
2. We can allow particles to jump anywhere, if they are aiming for some $x \in \mathbb{R}^d \setminus \Omega$,
they are "stopped" at a corresponding point on $\partial\Omega$.
3. We do not allow particles to jump outside the domain.
If they want to, they must "roll the dice" again from their current position.

These approaches lead to three different operators

1. Restricted fractional Laplacian

$$(-\Delta)^s_{\mathrm{RFL}}[u](x) = C_1(d,s)\,\mathrm{P.V.}\int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y$$

2. Censored fractional Laplacian

$$(-\Delta)^s_{\mathrm{CFL}}[u](x) = C_2(d,s)\,\mathrm{P.V.}\int_{\Omega} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y$$

3. Spectral fractional Laplacian

$$(-\Delta)^s_{\mathrm{SFL}}[u](x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_\Omega}[u](x) - u(x)\right)\frac{\mathrm{d}t}{t^{1+s}}.$$

A simple way to check that they are different is the boundary behaviour of solutions for $f \in L^\infty_c(\Omega)$.

$$\begin{cases} (-\Delta)^s_{\mathrm{RFL}} u = f & \Omega, \\ u = 0 & \Omega^c. \end{cases} \tag{$\mathrm{P_{RFL}}$}$$

Through weak formulation in $H^s(\mathbb{R}^d)$ it is to see that a unique solution exists.
People in probability computed

$$u(x) = \int_\Omega \mathbb{G}_{\mathrm{RFL}}(x,y) f(y) \,\mathrm{d}y$$

where [Kulczycki 1997] showed

$$\mathbb{G}_{\mathrm{RFL}}(x,y) \asymp \frac{1}{|x-y|^{d-2s}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right)^s .$$

This implies, for $f \geq 0$

$$u(x) \geq c\delta(x)^s \int_{\mathbb{R}^d} f(y)\delta(y)^s \,\mathrm{d}y.$$

Theorem 23 [Ros-Oton and Serra 2014]

Let u be a solution of ($\mathrm{P_{RFL}}$).
If $f \in L^\infty(\Omega)$ then $u \in C^s(\mathbb{R}^d)$ and $\|u\|_{C^s(\mathbb{R}^d)} \leq C\|f\|_{L^\infty}$.
If $f \in C^\beta(\overline{\Omega})$ and $\beta, \beta+2s \notin \mathbb{Z}$ then $u/\delta^s \in C^{\beta+2s}(\overline{\Omega})$.

$$\begin{cases} (-\Delta)^s_{\mathrm{RFL}} v = 0 & \Omega, \\ v = g & \Omega^c. \end{cases} \tag{$\mathrm{D_{RFL}}$}$$

Through weak formulation in $H^s(\mathbb{R}^d)$ it is to see that a unique solution exists.
People in probability computed

$$v(x) = \int_{\Omega^c} \mathbb{P}_{\mathrm{RFL}}(x,y) g(y) \, \mathrm{d}y$$

where showed that

$$\mathbb{P}_{\mathrm{RFL}}(x,y) = -(-\Delta)^s \mathbb{G}_{\mathrm{RFL}}(x,y) \asymp \frac{\delta(x)^s}{\delta(y)^s (1+\delta(y)^s) |x-y|^d}$$

The details in the analytical setting can be found in [Abatangelo 2015] .

In [Abatangelo 2015] the authors also showed that the following problem

$$\begin{cases} (-\Delta)^s_{\mathrm{RFL}} w = 0 & \Omega, \\ w = 0 & \Omega^c, \\ \displaystyle\lim_{x\to z} \frac{w(x)}{\delta(x)^{s-1}} = h(z) & z \in \partial\Omega \end{cases} \tag{$\mathrm{M}_{\mathrm{RFL}}$}$$

for any $h \in L^\infty(\partial\Omega)$.

In fact, the solution is unique and

$$w(x) = \int_{\partial\Omega} \mathbb{M}_{\mathrm{RFL}}(x,z) h(z) \, \mathrm{d}S_z ,$$

and

$$\mathbb{M}_{\mathrm{RFL}}(x,z) = \lim_{\substack{y\to z \\ y\in\Omega}} \frac{\mathbb{G}_{\mathrm{RFL}}(x,y)}{\delta(y)^s} .$$

Censored fractional Laplacian for $s > \frac{1}{2}$

$$\begin{cases} (-\Delta)^s_{\mathrm{CFL}} u = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \qquad (\mathrm{P}_{\mathrm{CFL}})$$

Through weak formulation in $H^s_0(\Omega)$ it is to see that a unique solution exists.
People in probability computed

$$u(x) = \int_\Omega \mathbb{G}_{\mathrm{CFL}}(x,y) f(y)\, \mathrm{d}y$$

where [Chen and Kim 2002] showed

$$\mathbb{G}_{\mathrm{CFL}}(x,y) \asymp \frac{1}{|x-y|^{d-2s}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right)^{2s-1}.$$

This implies, for $f \ge 0$

$$u(x) \ge c\delta(x)^{2s-1} \int_{\mathbb{R}^d} f(y)\delta(y)^{2s-1}\, \mathrm{d}y.$$

We also have

$$f \in L^\infty(\Omega) \implies \|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty},$$
$$f \in L^\infty_c(\Omega) \implies |u(x)| \le C_f \delta(x)^{2s-1}.$$

[Chen 2018] showed that if $u \in L^\infty(\Omega)$ implies $u \in C^\theta(K)$ for $\theta \in (0, 2s)$.

$$\begin{cases} (-\Delta)^s_{\mathrm{CFL}} w = 0 & \Omega, \\ w = h & \partial\Omega. \end{cases} \tag{$\mathrm{M}_{\mathrm{CFL}}$}$$

Existence and uniqueness for this problem was proved in [Chen and Kim 2002].

In [Abatangelo, GC, and Vázquez 2019], the authors prove that

$$u(x) = \int_{\partial\Omega} \mathbb{M}_{\mathrm{CFL}}(x,z) h(z) \, \mathrm{d}S_z$$

where

$$\mathbb{M}_{\mathrm{CFL}}(x,z) \asymp \lim_{\substack{y \to z \\ y \in \Omega}} \frac{\mathbb{G}_{\mathrm{RFL}}(x,y)}{\delta(y)^{2s-1}}.$$

Since $-\Delta_\Omega$ admits a spectral decomposition (λ_i, φ_i), the heat semigroup is written

$$e^{t\Delta_\Omega}[u](x) = \sum_{i=1}^\infty e^{-\lambda_i t} \left(\int_\Omega u(y)\varphi_i(y)\,\mathrm{d}y \right) \varphi_i(x).$$

Hence

$$(-\Delta)^s_{\mathrm{SFL}}[u] = \sum_{i=1}^\infty \lambda_i^s \left(\int_\Omega u(y)\varphi_i(y)\,\mathrm{d}y \right) \varphi_i(x).$$

Therefore $(-\Delta)^s_{\mathrm{SFL}}$ has spectral decomposition (λ_i^s, φ_i).
The boundary value is taken only in the operational sense.

We have that

$$\mathbb{G}_{\mathrm{SFL}}(x,y) = \sum_{i=1}^\infty \lambda_i^s \varphi_i(x)\varphi_i(y) \asymp \frac{1}{|x-y|^{d-2s}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right).$$

We can also set, for any $h \in L^\infty(\partial\Omega)$

$$\begin{cases} (-\Delta)^s_{\mathrm{SFL}} w = 0 & \Omega, \\ \displaystyle\lim_{x\to z} \frac{w(x)}{\delta(x)^{2(s-1)}} = h(z) & z \in \partial\Omega \end{cases} \tag{$\mathrm{M}_{\mathrm{SFL}}$}$$

[Bonforte, Sire, and Vázquez 2015] the authors introduce the idea of looking at

$$\begin{cases} Lu = f & \Omega, \\ u = 0 & \partial\Omega \text{ or } \Omega^c \end{cases}$$

by looking only at the the Green operator.
Assume its symmetric and, for some $\gamma \in [0,1]$,

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{d-2s}} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right)^\gamma .$$

It is easy to see that $f \in L^\infty_c(\Omega) \implies |u| \le C\delta^\gamma$

[Bonforte, Figalli, and Vázquez 2018]: The operator admits a spectral decomposition.

[Abatangelo, GC, and Vázquez 2019]:

1. Under suitable assumptions on $\mathbb{G}$,
   $f \in L^\infty_c(\Omega) \implies \exists D_\gamma[u](z) = \lim_{x \to z \in \partial\Omega} \frac{u(x)}{\delta(x)^\gamma}$.
2. There is a notion well-posed notion of weak-dual solution for $f \in L^1(\Omega, \delta^\gamma)$
   $$\int_\Omega u\psi = \int_\Omega f \mathcal{G}[\psi], \qquad \forall \psi \in L^\infty_c(\Omega).$$
3. $f \geq 0 \implies u(x) \geq c\delta(x)^\gamma \int_\Omega f(y)\delta(y)^\gamma \, \mathrm{d}y$, so this is the optimal set of data.
4. As $f_n \geq 0$ concentrate to the boundary to $h \in L^1(\partial\Omega)$, with $\int_\Omega f_n \delta^\gamma$ fixed, $u_n \to w$, we construct solutions of
   $$\int_\Omega w\psi = \int_{\partial\Omega} h D_\gamma \mathcal{G}[\psi], \qquad \forall \psi \in L^\infty_c(\Omega).$$
5. When $h = 1$ we have $w^\star \asymp \delta^{\min\{2s-\gamma-1, \gamma\}}$ (unless $2s - \gamma - 1 = \gamma$).
6. We construct a Martin kernel of solutions of this problem
   $$w(x) = \int_{\partial\Omega} \mathbb{M}(x,z) h(z) \, \mathrm{d}z, \qquad \mathbb{M}(x,z) = \lim_{y \to z} \frac{\mathbb{G}(x,y)}{w^\star(y)}$$

Following this same approach we can study the associated parabolic problems

$$\begin{cases} u_t + Lu = f & (0,T) \times \Omega, \\ u = 0 & \Omega^c \text{ (if applicable)}, \\ u/w^\star = h & \partial\Omega. \end{cases}$$

[Chan, Gómez-Castro, and Vázquez 2020]:

1. The information of the Green kernel is sufficient for a Weyl's law on the growth of the eigenvalues
2. A spectral decomposition allows to construct a heat kernel
3. There is a corresponding Duhamel formula
4. Regular solutions can be concentrated on the boundary to construct singular solutions

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