Aggregation–Diffusion Equations: concentration vs simplification

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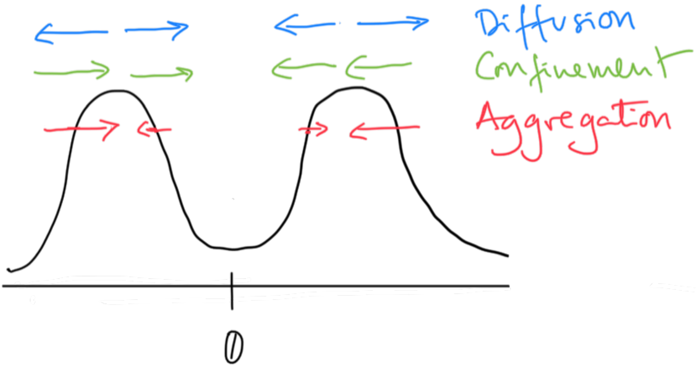
Diffusion
Confinement
Aggregation
0

The aim of this talk is to explain the modeling and theory behind the following model for aggregation-diffusion phenomena:

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla (\underbrace{U'(\rho)}_{\text{Diffusion}} + \underbrace{V}_{\text{Confinement}} + \underbrace{W * \rho}_{\text{Aggregation}})\right) \tag{ADE}$$

We will discuss the range of power-type aggregation and diffusion

$$U'(\rho) = \tfrac{m}{m-1}\rho^{m-1}, \qquad V(x) = \frac{|x|^\alpha}{\alpha}, \qquad \text{and} \qquad W(x) = \frac{|x|^\lambda}{\lambda}.$$

If V, W are bounded below, we can always assume $V, W \geq 0$.

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Conservation equation. Let ρ be a density and $\omega \subset \mathbb{R}^d$ any control volume, if $\mathbf{j}$ is the out-going flux

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho \, \mathrm{d}x = - \int_{\partial\omega} \mathbf{j} \cdot \mathbf{n} \, \mathrm{d}S = - \int_\omega \operatorname{div} \mathbf{j} \, \mathrm{d}x$$

Linear Darcy's law: flux opposing the gradient $\mathbf{j} = -\nabla\rho$ yields

$$\frac{\partial \rho}{\partial t} = \Delta \rho \tag{HE}$$

The confinement can be added as a drift $\mathbf{j} = -\nabla\rho - \rho\nabla V$.

Non-linear Darcy's law: $\mathbf{j} = -\nabla\varphi(\rho)$ for some non-decreasing $\varphi : \mathbb{R} \to \mathbb{R}$

$$\frac{\partial \rho}{\partial t} = \Delta \varphi(\rho). \tag{DE}$$

When $\varphi(\rho) = \rho^m$ for $m > 0$ this is called Porous Medium Equation [Vázquez 2006].
Notice $\Delta\varphi(\rho) = \operatorname{div}(\varphi'(\rho)\nabla\rho)$ so $\quad U''(\rho) = \frac{\varphi'(\rho)}{\rho}$.

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Consider N with positions X_i of equal masses $1/N$ and the attracting/repulsive system[1]

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \underbrace{-\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{N} \nabla W(X_i - X_j)}_{\text{Aggregation}} \underbrace{-\frac{1}{N}\nabla V(X_i)}_{\text{Confinement}}, \qquad i = 1, \cdots, N$$

The empirical distribution is defined as $\mu_t^N = \sum_{j=1}^{N} \frac{1}{N} \delta_{X_j(t)}$.

In the sense of distributions, μ^N solves the **Aggregation Equation**

$$\partial_t \mu = \mathrm{div}(\mu \nabla (W * \mu + V)) \tag{AE}$$

Diffusion can added to the particle system by introducing noise Details .

[1] Assume $\nabla W(0) = 0$

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Joining the many particle approximation with the Porous Medium diffusion:

$$\frac{\partial \rho}{\partial t} = \mathrm{div}\left(\rho \nabla (U'(\rho) + V + W * \rho)\right) \tag{ADE}$$

Some famous examples

Model	U	V	W
Heat Equation	$\rho \log \rho$	0	0
Porous Medium Equation $m \neq 1$	$\frac{1}{m-1} \rho^m$	0	0
Fokker-Planck	$\rho \log \rho$	$\frac{1}{2}\|x\|^2$	0
Keller-Segel ($d = 2$)	$\rho \log \rho$	0	$-\frac{1}{2\pi} \log \|x\|$
Swarming / Herding	0	0	$\frac{1}{a}\|x\|^a - \frac{1}{b}\|x\|^b$

Conservation?

In conservation laws, we expect $\int_{\mathbb{R}^d} \rho(t) = \int_{\mathbb{R}^d} \rho_0$
(i.e. $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$, we expect $\rho(t) \in \mathcal{P}(\mathbb{R}^d)$)

A direct computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \lim_{R\to\infty} \int_{B_R} \rho \,\mathrm{d}x = \lim_{R\to\infty} \int_{\partial B_R} j \,\frac{x}{|x|} \,\mathrm{d}S \overset{?}{=} 0.$$

Sometimes mass is not conserved, and we will give an example later.

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It admits a solution

$$K(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

- $K(t,\cdot) \to \delta_0$ as $t \to 0^+$
- $\|K(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1.$
- For any x, $K(t,x) \to 0$ as $t \to \infty$
- Any solution satisfies

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0 \qquad \text{as } t \to \infty.$$

This is known as **asymptotic simplication**.

The range $m \in \left((\frac{d-2}{d})_+, 1 \right) \cup (1, +\infty)$:

Admits the Barenblatt solution (see [Vázquez 2006])

$$B(t,x) = t^{-\alpha} \left(C_1 - C_2 |x|^2 t^{\frac{2\alpha}{d}} \right)_+^{\frac{1}{m-1}}, \qquad \alpha = \frac{d}{d(m-1)-2}$$

That has the expect properties:

- $B(0^+, x) = \delta_0(x)$,
- $B(t,x) \to 0$ as $t \to \infty$,
- $\|B(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$, and
- Asymptotic simplication

$$\|\rho(t,\cdot) - B(t,\cdot)\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } t \to \infty.$$

For $0 < m < \frac{d-2}{d}$ there are two surprising facts:

- There is **finite-time extinction**

  If $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$ Details

$$\|\rho(t)\|_{L^q} \searrow 0, \qquad \text{as } t \nearrow T^* < \infty.$$

- There is always **infinite-time** total mass loss

$$\|\rho(t)\|_{L^1} \to 0$$

  . Details

- [Brezis and Friedman 1983] proved that δ_0 does not diffuse:
  if we take a sequence $\rho_j(0^+, \cdot) \to \delta_0$, the associated solution $\rho_j(t) \to \delta_0$.

Notice that the heat kernel is **self-similar**

$$K(t,x) = (2t)^{-\frac{d}{2}} G(-\tfrac{x}{\sqrt{2t}}), \qquad G(y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp(-\tfrac{|y|^2}{2})$$

So send K to G: $\tau = \log \sqrt{2t+1}$, $y = \frac{x}{\sqrt{2t+1}}$ and $u(\tau, y) = e^{d\tau} \rho(t, x)$

Applying the change of variable to the heat equation we recover the **Fokker-Planck equation**

$$\begin{aligned} \partial_\tau u &= \Delta_y u + \operatorname{div}(yu) \\ &= \operatorname{div}(u \nabla_y (\log u + \tfrac{|y|^2}{2})). \end{aligned}$$

Clearly, G is a stationary solution.
Furthermore, G is an **asymptotic profile** for the equation:

$$\|u(t,\cdot) - G\|_{L^1} \to 0, \qquad \text{as } t \to \infty.$$

A similar approach works for the Porous Medium Equation, where the profile is B.

The Keller-Segel proposed a model of cell migration by chemotaxis given by

$$\begin{cases} \partial_t \rho = \Delta\rho + \mathrm{div}(\rho\nabla v), \\ -\Delta v = u. \end{cases} \qquad M = \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x$$

For $d \geq 2$ we can write $v = W * u$ for W the Newtonian potential.

There exists $M^* > 0$ such that

- If $M < M^*$ solutions are global-in-time.
- If $M > M^*$ there is finite-time blow-up. And $\rho(T^*) = M\delta_0$.

Finite time blow-up

The Keller-Segel proposed a model of cell migration by chemotaxis given by

$$\begin{cases} \partial_t \rho = \Delta \rho + \operatorname{div}(\rho \nabla v), \\ -\Delta v = u. \end{cases} \qquad M = \int_{\mathbb{R}^d} \rho_0(x)\, \mathrm{d}x$$

Case $d = 2$

- [Jäger and Luckhaus 1992] There exists M^* s.t. $M > M^*$, then $\rho(T^*)$ contains a Dirac delta. They work in Ω bounded (+no-flux condition).
- [Herrero and Velázquez 1996]: For $\mathbb{R}^d$ we have $M^* = 8\pi$. Explicit constructions.
- [Dolbeault and Perthame 2004]: $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2 \rho(t,x)\, \mathrm{d}x = 4M \left(1 - \frac{M}{8\pi}\right)$
  If $M > M^*$, then $\rho(T^*) = M\delta_0$
- [Blanchet, Dolbeault, and Perthame 2006]: For $M < M^*$ global existence.

Case $d \geq 3$

- [Blanchet, Carrillo, and Laurençot 2009]:
  There is a functional $\mathcal{F}$ decaying along trajectories and
  $$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2 \rho(t,x)\, \mathrm{d}x = 2(d-2)\mathcal{F}[\rho(t,\cdot)] \leq 2(d-2)\mathcal{F}[\rho_0].$$
  When $M > M^*$ there exist $\|\rho_0\|_{L^1} = M$ such that $\mathcal{F}[\rho_0] < 0$.

Main question

For

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla (U'(\rho) + V + W * \rho)\right) \qquad \text{(ADE)}$$

can we classify characterise ρ_∞ such that

$$\rho(t) \to \rho_\infty$$

in terms of general U, V, W?

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Gradient flow in $\mathbb{R}^d$

Let $F : \mathbb{R}^d \to \mathbb{R}$. Imagine we look for $\operatorname{argmin} F$.

We call **gradient flow** of F the flow of the ODE $$\frac{\mathrm{d}X}{\mathrm{d}t} = -\nabla F(X(t))$$

If F is strictly convex, for any $X(0)$ we have $X(t) \to X_\infty = \operatorname{argmin} F$.

If $D^2 F \geq \lambda I$ then $|X(t) - X_\infty| \leq e^{-\lambda t}|X_0 - X_\infty|$.

The Heat Equation as a gradient flow in L^2

Let $\mathcal{F} : H^1(\mathbb{R}^d) \to \mathbb{R}$ be defined as $\mathcal{F}[\rho] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \rho|^2$

Formally

$$\nabla_{L^2} \mathcal{F}[\rho_0] = \frac{\delta \mathcal{F}}{\delta \rho}[\rho_0] = -\Delta \rho_0$$

Remark

We can rewrite the Heat Equation

$$\frac{\partial \rho}{\partial t} = -\nabla_{L^2} \mathcal{F}[\rho(t)], \quad \text{where } \mathcal{F}[\rho] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \rho|^2 \tag{HE}$$

$\mathcal{F}$ is strictly convex in $L^2(\mathbb{R}^d)$. Naturally, $\rho(t) \to 0$ which is the "minimiser" of $\mathcal{F}$.

In general, the $\nabla_{L^2} \mathcal{F}$ is given by the Euler-Lagrange equations ▸ Details

Our equations are "nice" in 2-Wasserstein space ($\mathcal{P}_2$). ▸ Details

For $\mathcal{F} : L^1 \cap \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ formally speaking ▸ Details

$$\nabla_{d_2} \mathcal{F}[\rho] = -\operatorname{div}\left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho}\right)$$

Remark

If $W(x) = W(-x)$, we can formally rewrite the Aggregation-Diffusion problem as[2]

$$\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)], \quad \text{where } \mathcal{F}[\rho] = \int_{\mathbb{R}^d} \left(U(\rho) + V\rho + \tfrac{1}{2}\rho(W * \rho) \right). \tag{ADE}$$

Formally, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^d} \rho \left| \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho] \right|^2$. This is called *energy dissipation* estimate.

The precise definition of solution is the notion of *curves of maximal slope*. ▸ Details

[2] Due to the convolution, $\mathcal{F}$ is non-local and $\mathcal{F}[\rho] \neq \int_{\mathbb{R}^d} F(x, \rho(x))\,\mathrm{d}x$. $\frac{\delta F}{\delta \rho}$ can be computed directly

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The extension of convexity in $\mathbb{R}^d$ that is suitable in $\mathcal{P}_2$ is **displacement convexity** (see [McCann 1997])

There is a suitable theory for gradient flow of $\mathcal{F}$ in d_2 (see [Ambrosio, Gigli, and Savare 2005])

In fact, as $t \to \infty$ we have

$$\mathcal{F}[\rho(t)] \searrow \inf_{\rho \in \mathcal{P}_2 \cap L^1} \mathcal{F}.$$

Under stronger hypothesis, if $\rho_\infty = \operatorname{argmin} \mathcal{F}$ then

$$d_p(\rho(t), \rho_\infty) \to 0.$$

Due to the estimate above, at an minimiser ρ_∞, we have

$$\rho_\infty \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = 0.$$

Either $\rho_\infty = 0$ (as in PME), or $\frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = C$ (over open sets).

The free energy for (ADE)

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} \left(U(\rho) + V\rho + \frac{1}{2}\rho(W * \rho) \right) .$$

In some cases it is displacement convex (see [Carrillo and Slepčev 2009]):
$m \geq \frac{d-1}{d}$ and V, W convex.

Recall for $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ we can leave $\mathcal{P}_2$.

When $\inf \mathcal{F} = -\infty$, then we do not expect an asymptotic equilibrium.
(maybe intermediate asymptotics)

Actually, we need to consider the extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$, which we denote $\widetilde{\mathcal{F}}$
(see [Demengel and Temam 1986]) ▸ Details

If $\mu_\infty \in \operatorname{argmin}_{\|\mu\|=M} \widetilde{\mathcal{F}}[\mu]$, we expect it to be a local attractor (no guarantee).

The first variation is: $\dfrac{\delta \mathcal{F}}{\delta \rho} = U'(\rho) + V + W * \rho.$

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The free energy for Keller-Segel is

$$\mathcal{F}[\rho] = \underbrace{\int_{\mathbb{R}^2} \rho \log \rho}_{\mathcal{S}[\rho]} + \underbrace{\frac{-1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho(x)\rho(y) \log|x-y| \, \mathrm{d}x \, \mathrm{d}y}_{\mathcal{I}[\rho]} .$$

To see whether diffused or concentrated is energy beneficial: $\rho_\lambda(x) = \lambda^2 \rho_1(\lambda x)$.

Then, letting $M = \int_{\mathbb{R}^2} \rho_1$ we have

$$\mathcal{S}[\rho_\lambda] = \mathcal{S}[\rho_1] + 2M \log \lambda$$

$$\mathcal{I}[\rho_\lambda] = \mathcal{I}[\rho_1] - \frac{M^2}{2\pi} \log \lambda$$

Eventually

$$\mathcal{F}[\rho_\lambda] = \mathcal{F}[\rho_1] + 2M \log \lambda \left(1 - \frac{M}{8\pi} \right)$$

This gives the intuition that for $M > 8\pi$ then δ_0 (i.e. $\lambda \to \infty$) is energy beneficial.

Free-energy minimisation for (ADE) when $V = 0$

Existence and non-existence of δ_0

Minimisation for $U = \frac{m}{m-1}\rho^m$, $V = 0$, and $W(x) = |x|^\lambda/\lambda$:

- [Carrillo, Hittmeir, Volzone, and Yao 2019]:
  Any minimiser is $\mu_\infty = \rho_\infty + M\delta_0$ with ρ_∞ radially symmetric
- [Carrillo, Delgadino, Dolbeault, Frank, and Hoffmann 2019]:
  $\lambda > 0$ and $m \in (0,1)$
  - $\mathcal{F}$ is bounded below iff $m \in (\frac{d}{d+\lambda}, 1)$
  - If $m > \frac{d}{d+\lambda}$ there exists a minimiser of the form $\mu_\infty = \rho_\infty + M\delta_0$
  - If $\lambda \in [2,4]$ or $\lambda \geq 1$ and $m \geq 1 - \frac{1}{d}$, then the global minimiser is unique (up to translation).
  - $M = 0$ if $\lambda \in (0, 2 + \frac{4}{(N-2)_+})$ and $m \in (\frac{d}{d+\lambda}, 1)$
- [Carrillo, Delgadino, Frank, and Lewin 2020]:
  - If $\lambda = 4$ and $d \leq 5$ then $M = 0$.
  - If $\lambda = 4$ and $d \geq 6$ then $M = 0$ if and only if $m \geq \frac{d-2}{d+4}\left(1 + \frac{4}{3d}\right)$.
  - Numerical results for $\lambda = 2k$

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The Euler-Lagrange equation is

$$-\frac{m}{1-m}\rho^{m-1} + V = -C.$$

We deduce that the minimising profiles are

$$\rho_{V+h} = \left(\frac{1-m}{m}(V+h)\right)^{-\frac{1}{1-m}}$$

By standard minimisation arguments, we show that

$$\mu_\infty = \begin{cases} \rho_{V+h} & \text{if } \exists h \geq 0 \text{ such that } \|\rho_{V+h}\|_{L^1} = \|\rho_0\|_{L^1}, \\ \rho_V + \Big(\|\rho_0\|_{L^1} - \|\rho_V\|_{L^1}\Big)\delta_0 & \text{if } \|\rho_0\|_{L^1} > \|\rho_V\|_{L^1} \end{cases}$$

The question is

$$\rho(t) \to \rho_\infty?$$

Outside the displacement convex range, we have to go back to PDE methods.

[Cao and Li 2020]: If $\rho_{V+h_1} \le \rho_0 \le \rho_{V+h_2}$ with $h_1, h_2 > 0$ then L^∞_{loc} convergence to

$$\rho_\infty = \rho_{V+h}, \qquad \text{for some } h > 0.$$

[Carrillo, G-C, and Vázquez 2021]:
Under some technical hypothesis on V regular, assume that $\|\rho_V\|_{L^1} < 1$.
Then $\exists \rho_0 \in L^1(\mathbb{R}^d)$ such that $\forall r > 0$

$$M(t,r) = \int_{B_r} \rho(t,x)\,\mathrm{d}x \nearrow \int_{B_r} \rho_V(x)\,\mathrm{d}x + \underbrace{1 - \|\rho_V\|_{L^1}}_{\text{a Dirac!}} \qquad \text{with } a > 0 \text{ as } t \to \infty.$$

The key idea is that M satisfies a Hamilton-Jacobi type equation.

▸ Sketch of Proof

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We start by a classical observation.

Remark

If $W \in L^\infty(\mathbb{R}^d)$, then there are no finite-mass steady states.

If $W \in L^\infty(\mathbb{R}^d)$ and $\rho \in L^1(\mathbb{R}^d)$, then $W * \rho \in L^\infty(\mathbb{R}^d)$.
The Euler-Lagrange equation is $\log \rho + W * \rho = c$, so

$$\rho = e^c e^{W*\rho} \geq e^c e^{-\|W*\rho\|_{L^\infty}} > 0.$$

Therefore, in this range we always expect diffusion.

In the case of linear diffusion

$$\partial_t \rho = \Delta\rho + \mathrm{div}(\rho \nabla W * \rho).$$

[Cañizo, Carrillo, and Schonbek 2012]: for small W:

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0 \qquad \text{as } t \to \infty. \qquad (\star)$$

where K is the heat kernel.

Theorem [Carrillo, G-C, Yao, and Zeng 2021] ▸ Sketch of proof

Let $n \geq 2$, and assume $W(x) = W(-x)$

- $W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$
- $\nabla W \in L^{n-\varepsilon}(\mathbb{R}^d)$
- $\Delta W \in L^{\frac{n}{2}}(\mathbb{R}^d)$ (and also $\Delta W \in L^{\frac{n}{2}-\varepsilon}(\mathbb{R}^d)$ if $n \geq 3$)

Then $(\star)$.

Notice that this hypothesis work for $W(x) \sim |x|^{-\varepsilon}$ for any $\varepsilon > 0$,
but not for the critical case $W(x) \sim \log|x|$.

Question that arise:

1. Does this idea work for any $\rho_0 \in L^1$?
2. Does it work for some classes $W \neq 0$.

The answer seems positive for some W.
PhD plan of Alejandro Fernández-Jiménez (U. Oxford).

Thank you for your attention!

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Consider an stochastic particle jumping over the mesh $\{\ldots, -h, 0, h, 2h, \ldots\}$ $(h > 0)$.
Let X_n be the position after n jumps. Assume the jump probabilities are

$$\mathbb{P}(X_{n+1} = jh \mid X_n = ih) = \begin{cases} \frac{1}{2} & \text{if } |i-j| = 1, \\ 0 & \text{otherwise} \end{cases}$$

Define $U_j^n = P(X_n = hj)$ and assume the initial distribution U_j^0 is given.
Then $U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n)$ or, for $\tau = h^2$

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \frac{1}{2} \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}.$$

This is a classical discretisation of the stochastic version of (HE): $\partial_t \rho = \frac{1}{2} \Delta \rho$.

The time continuous extension of X_n version is the Wiener process $X_t = W_t$.
This gives rise to the intuition (which has to be understood in terms of the Îto calculus)

$$dX_t = dW_t$$

Consider 1 particle. Using a similar arguments, for the stochastic equation

$$\mathrm{d}X_t = \underbrace{\mu(t, X_t)}_{\text{drift}}\,\mathrm{d}t + \underbrace{\sigma(t, X_t)\,\mathrm{d}W_t}_{\text{diffusion}}$$

its probability density ρ is the solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t}(t, x) = -\operatorname{div}(\mu(t, x)\rho(t, x)) + \Delta(D(t, x)\rho(t, x))$$

where $D = \frac{\sigma^2}{2}$.

Imagine now we have N stochastic particles at positions $X_1(t), \cdots, X_N(t)$. We assume they have equal mass.

Recall the empirical measure $\mu_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_j(t)}$

Assume the evolution of the particles is given by the system of SODEs

$$\mathrm{d}X_i = -\frac{1}{N} \sum_{j \neq i} \nabla W(X_i - X_j) - \frac{1}{N} \nabla V(X_i) + \sqrt{2D}\, \mathrm{d}W_t^i$$

The limit as $N \to \infty$ is the solution of

$$\partial_t \rho = \mathrm{div}(\rho \nabla (W * \rho + V)) + D\Delta\rho.$$

This corresponds to $U(\rho) = D\rho \log \rho$.

Mean-Field Approximation: As $N \to \infty$

$$\mu_0^N \to \rho_0 \text{ in the tight topology} \implies \mu_t^N \to \rho(t) \text{ in law for a.e. } t > 0.$$

For the details see, e.g., [Jabin and Wang 2017].

[2] Convergence in law: pointwise convergence of distribution functions at continuity points of the limit

Let $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ and $d \geq 3$ and $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{q} \int_{\mathbb{R}^d} \rho^q \overset{\text{PDE}}{=} -C \int_{\mathbb{R}^d} |\nabla \rho^{\frac{m+q}{2}}|^2 \overset{\text{Sobolev}}{\leq} -C \Big(\int_{\mathbb{R}^d} \rho^{\frac{m+q}{2} 2^*} \Big)^{\frac{1}{2^*}}$$

where $2^* = \frac{1}{2} - \frac{1}{d}$.
The equation $\frac{\mathrm{d}}{\mathrm{d}t} X = -C X^\alpha$ where $\alpha < 1$ has finite time extinction.

PME: Asymptotic loss of mass

Go back

Take $\varepsilon > 0$, taking $\rho_0^\varepsilon \in L^q$ with $\|\rho_0 - \rho_0^\varepsilon\|_{L^1} \leq \varepsilon$. For $t \geq T_\varepsilon^*$

$$\begin{aligned} \|\rho(t)\|_{L^1} &\leq \|\rho(t) - \rho^\varepsilon(t)\|_{L^1} + \|\rho^\varepsilon(t)\|_{L^1} \\ &\leq \|\rho_0 - \rho_0^\varepsilon\|_{L^1} + 0 \\ &\leq \varepsilon. \end{aligned}$$

$\mathcal{P}_2$ is not a vector space, so there we are not using the intrinsic notion of Fréchet gradient.
The correct notion is **Fréchet subdifferentials** (we will not define it here).
Also, we can see $\mathcal{P}_2$ inside the space of measures.
Fix $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then, the tangent space is given by

$$\mathrm{Tan}_{\rho_0}\mathcal{P}_2(\mathbb{R}^d) = \left\{ \xi : \exists \zeta_n \in C_c(\mathbb{R}^d, \mathbb{R}) \text{ s.t. } \int_{\mathbb{R}^d} |\xi - \nabla \zeta_n|^2 \, \mathrm{d}\rho_0 \to 0 \right\}$$

Take $\xi = \nabla \zeta$ with $\zeta \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Then, by [Ambrosio, Gigli, and Savare 2005, Lemma 5.5.3]

$$\rho_\varepsilon = (1_{\mathbb{R}^d} + \varepsilon \xi)_{\#}\rho_0 = \frac{\rho_0}{\det(1_{\mathbb{R}^d} + \varepsilon \nabla \xi)} \circ (1_{\mathbb{R}^d} + \varepsilon \xi)^{-1}$$

The map $(x, \varepsilon) \mapsto \rho_\varepsilon(x)$ is C^2 and

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = \rho_0, \qquad \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon = 0} \rho_\varepsilon = -\operatorname{div}(\rho \xi).$$

For ε small enough $1_{\mathbb{R}^d} + \varepsilon \nabla \zeta$ is an optimal transport map, so ρ_ε is a constant-speed geodesic.
Hence, using standard variation formulae (see [Giaquinta and Hildebrandt 1996])

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}[\rho_\varepsilon] - \mathcal{F}[\rho_0]}{\varepsilon} = -\int_{\mathbb{R}^d} \frac{\delta F}{\delta \rho}[\rho_0] \nabla \cdot (\rho \xi) = \int_{\mathbb{R}^d} \nabla \zeta \nabla \frac{\delta F}{\delta \rho}[\rho_0] \, \mathrm{d}\rho$$

This characterises $\nabla_{d_2} \mathcal{F} = -\operatorname{div}(\rho \nabla \frac{\delta F}{\delta \rho})$ in a broad distributional sense.

Let

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} F(x, \rho(x), \nabla\rho(x))\, \mathrm{d}x.$$

Expanding $F(x, s, \xi)$ in Taylor expansion yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\mathcal{F}[\rho_0 + \varepsilon\varphi] - \mathcal{F}[\rho_0]}{\varepsilon} &= \int_{\mathbb{R}^d} \left(\tfrac{\partial F}{\partial s}(x, \rho_0, \nabla\rho_0)\varphi + \tfrac{\partial F}{\partial \xi}(x, \rho_0, \nabla\rho_0) \cdot \nabla\varphi \right) \\ &= \int_{\mathbb{R}^d} \left(\tfrac{\partial F}{\partial s}(x, \rho_0, \nabla\rho_0) - \operatorname{div}\left[\tfrac{\partial F}{\partial \xi}(x, \rho_0, \nabla\rho_0) \right] \right) \varphi\end{aligned}$$

Thus

$$\nabla_{L^2}\mathcal{F}[\rho_0] = \frac{\delta\mathcal{F}}{\delta\rho}[\rho_0] = \frac{\partial F}{\partial s}[\rho_0] - \operatorname{div}\left(\frac{\partial F}{\partial \xi}[\rho_0] \right).$$

This is the Euler-Lagrange equation!

Given

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} f(\rho)\,\mathrm{d}x$$

where $f : \mathbb{R} \to \mathbb{R}$.

The questions is what is the natural lower semicontinuous extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$ with the weak-$\star$ topology.

Given a measure μ and mollifiers η_ε we define $\rho_\varepsilon = \mu * \eta_\varepsilon$.
For $|f(\xi)| \le C(1 + |\xi|)$ define

$$f_\infty(\xi) = \lim_{t\to\infty} \frac{f(t\xi)}{t}.$$

Since we can use the Lebesgue decomposition theorem $\mu = \rho\,\mathrm{d}x + \mu^s$, where ρ is the Radon-Nikodym derivative of μ. Then

$$\widetilde{F}[\mu] = \int_{\mathbb{R}^d} f(\rho)\,\mathrm{d}x + f_\infty(\mu^s).$$

The notion of $f_\infty(\mu^s)$ is tricky (but possible) to define.
If $f(s) = s^m$ with $m < 1$, then $f_\infty = 0$.

Curves of maximal slope

(see [Ambrosio, Gigli, and Savare 2005]) Go back

Mathematical Institute

Typically, $\frac{\partial \rho}{\partial t} = -\nabla_X \mathcal{F}[\rho(t)]$ for $X = L^2, H^1$ is satisfied in the dual sense.

The way in which $\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)]$ in rather tricky since $\mathcal{P}_2$ is not linear a space.

The main idea is the equivalence for $u : [0, T] \to \mathbb{R}^d$ that

$$u'(t) = -\nabla \mathcal{F}(u), \qquad \iff \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F} \circ u) = -|\nabla F(u)||u'| & \text{orientation} \\ |u'| = |\nabla \mathcal{F}(u)| & \text{norm} \end{cases}$$

We define the metric slopes

$$|\mu'|(t) = \limsup_{h \to 0} \frac{d_2(\mu(t+h), \mu(t))}{h}, \qquad |\partial \mathcal{F}|[\mu] = \limsup_{\nu \to \mu} \frac{(\mathcal{F}[\mu] - \mathcal{F}[\nu])_+}{d_2(\mu, \nu)}$$

Definition 1 Maximal slope curve

A locally abs. cont. curve $t \mapsto \mu(t) \in \mathcal{P}_2(\mathbb{R}^d)$ such that $t \mapsto \mathcal{F}[\mu(t)]$ is abs. cont. and

$$\frac{1}{2} \int_s^t |\mu'|^2(r) \, \mathrm{d}r + \frac{1}{2} \int_s^t |\partial \mathcal{F}|^2[\mu(r)] \, \mathrm{d}r \le \mathcal{F}[\mu(s)] - \mathcal{F}[\mu(t)] \qquad \forall 0 \le s < t \le T$$

Given a radially decreasing $\rho \geq 0$, $\rho^q \in L^1(B_R)$ for some $q > 0$ (for any $R \leq \infty$), using and old trick of Lieb's (see [Lieb 1977; Lieb 1983]) we get, for $|x| \leq R$,

$$\int_{B_R} \rho^q \,\mathrm{d}x = n\omega_n \int_0^R \rho(r)^q r^{n-1} \,\mathrm{d}r \geq n\omega_n \int_0^{|x|} \rho(r)^q r^{n-1} \,\mathrm{d}r \geq n\omega_n \rho(x)^q \int_0^{|x|} r^{n-1} \,\mathrm{d}r.$$

Hence, we deduce the point-wise estimate

$$\rho(x) \leq \left(\frac{\int_{B_R} \rho^q}{n\omega_n |x|^n} \right)^{\frac{1}{q}} . \tag{1}$$

It is easy to see that (1) is not sharp. However, it is useful to prove tightness for sets of probability measures.

For (DE) **entropy solutions**:

$$\rho_0 \in L^1 \implies \exists! \rho \in C([0,+\infty); L^1(\mathbb{R}^d))$$

(see, e.g. [Kružkov 1970; Carrillo 1999])

Keller-Segel with $M > M^*$:

$$\rho(t) \longrightarrow M\delta_0 \qquad \text{as } t \nearrow T < \infty.$$

The L^1 framework is not enough!

Total variation: $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^d)$.

However, if $a \neq b$ then $\|\delta_a - \delta_b\|_{\mathcal{M}} = 2$.

For the particle system $t \mapsto \mu_t^N$ is not $C\Big([0,T];(\mathcal{M}(\mathbb{R}^d), \|\cdot\|_{\mathcal{M}})\Big)$.

We want to construct a distance between measures such that

$$d(\delta_a, \delta_b) = |a - b|.$$

The Wasserstein distance

Go back

Given $\mu, \nu \in \mathcal{P}(X)$, taking plans between μ and ν:

$$\Pi(\mu,\nu) = \Big\{\pi \in \mathcal{P}(X \times Y) : \quad \pi(A \times Y) = \mu(A), \quad \pi(X \times B) = \nu(B)\Big\}.$$

we define the p-Wasserstein distance

$$d_p(\mu,\nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} |x-y|^p \, \mathrm{d}\pi(x,y) \right)^{\frac{1}{p}}$$

The correct space to work with this distance is

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \, \mathrm{d}\mu(x) \right\}$$

We endow $\mathcal{P}_p(\mathbb{R}^d)$ with the distance d_p.

Sketch of the proof of [Carrillo, G-C, and Vázquez 2021]

Go back

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For simplicity, we restrict to first bounded domains

$$\begin{cases} \partial_t u = \Delta \rho^m + \operatorname{div}(u \nabla V) & x \in \Omega, t > 0 \\ \partial_n u = \partial_n V = 0 & \partial\Omega. \end{cases}$$

For $m \in (0,1)$, $V \in W^{1,\infty}$, then the solution is well-defined and regular for all $t > 0$.

If ρ is radially symmetric, the mass function

$$M(t,r) = \int_{B_r} \rho(t,x)\,\mathrm{d}x$$

can be studied as viscosity solution of a Hamilton-Jacobi type equation
(see [Crandall, Ishii, and Lions 1992]).

For viscosity solutions may properties are known: stability, C^α regularity, ...

We construct an explicit initial datum ρ_0 such that $\partial_t M \geq 0$.

In the limit $M(t,\cdot) \nearrow M_\infty$, a solution of the mass of Euler-Lagrange and $M_\infty(R_v) = 1$.

There are no solutions of sufficient mass. Therefore $M_\infty(0^+) = a > 0$.

Sketch of proof of [Carrillo, G-C, Yao, and Zeng 2021]

Go back

First, we prove well-posedness by Duhamel's formula and that, in rescaled variable

- If $\nabla W \in L^n(\mathbb{R}^d)$ then $\sup_{\tau\geq 1} \|\widetilde{\rho}(\tau,\cdot)\|_{H^1} < \infty$.
- If $n \geq 2$, $\nabla W \in L^n(\mathbb{R}^d)$ and $\Delta W \in L^{\frac{n}{2}}(\mathbb{R}^d)$ then $\sup_{\tau\geq 1} \|\widetilde{\rho}(\tau,\cdot)\|_{C^\alpha} < \infty$ (modulus of continuity arguments, e.g. [Kiselev, Nazarov, and Volberg 2007])

Using a smart change in variable $\mathcal{F}[\rho(t)] \leq -\frac{n}{2} \ln t + C(n, \|W\|_{L^\infty})$.

Thus, we prove that $\int_{\mathbb{R}^d} \widetilde{\rho}(\tau,y) |\log \widetilde{\rho}(\tau,y)| \,\mathrm{d}y \leq C$.

We study the L^1 relative entropy $E_1(\widetilde{\rho}\|G) = \int_{\mathbb{R}^d} \widetilde{\rho} \log \frac{\widetilde{\rho}}{G} \,\mathrm{d}y$.

Like for (HE), we can use logarithmic Sobolev inequality
to recover an Ordinary Differential Inequality for E_1
(where the terms from W are a controlled errors)

Lastly, we apply the Csiszar-Kullback inequality

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} = \|\widetilde{\rho}(t,\cdot) - G(\cdot)\|_{L^1} \leq 2\sqrt{E_1(\widetilde{\rho}\|G)} \to 0, \qquad \text{as } t \to \infty.$$