Aggregation–Diffusion Equations: concentration vs simplification

D. Gómez-Castro
Mathematical Institute
University of Oxford

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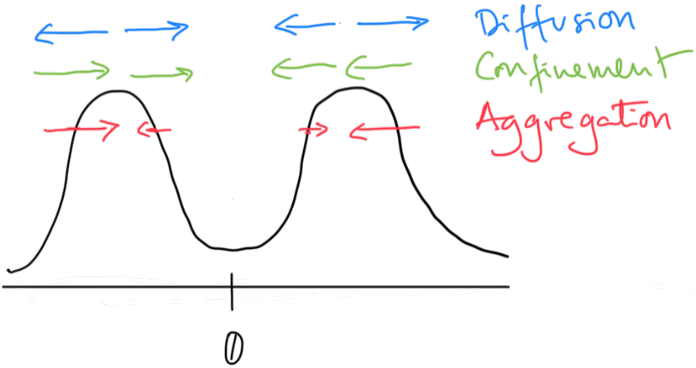

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The aim of this talk is to explain the modeling and theory behind the following model for aggregation-diffusion phenomena:

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla \big(\underbrace{U'(\rho)}_{\text{Diffusion}} + \underbrace{V}_{\text{Confinement}} + \underbrace{W * \rho}_{\text{Aggregation}} \big)\right) \tag{ADE}$$

We will discuss the range of power-type aggregation and diffusion

$$U'(\rho) = \tfrac{m}{m-1}\rho^{m-1}, \qquad V(x) = \frac{|x|^\alpha}{\alpha}, \qquad \text{and} \qquad W(x) = \frac{|x|^\lambda}{\lambda}.$$

If V, W are bounded below, we can always assume $V, W \geq 0$.

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Conservation equation. Let ρ be a density and $\omega \subset \mathbb{R}^d$ any control volume, if $\mathbf{j}$ is the out-going flux

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho \, \mathrm{d}x = -\int_{\partial\omega} \mathbf{j} \cdot \mathbf{n} \, \mathrm{d}S = -\int_\omega \operatorname{div} \mathbf{j} \, \mathrm{d}x$$

Linear Darcy's law: flux opposing the gradient $\mathbf{j} = -\nabla \rho$ yields

$$\frac{\partial \rho}{\partial t} = \Delta \rho \tag{HE}$$

The confinement can be added as a drift $\mathbf{j} = -\nabla \rho - \rho \nabla V$.

Non-linear Darcy's law: $\mathbf{j} = -\nabla \varphi(\rho)$ for some non-decreasing $\varphi : \mathbb{R} \to \mathbb{R}$

$$\frac{\partial \rho}{\partial t} = \Delta \varphi(\rho). \tag{DE}$$

When $\varphi(\rho) = \rho^m$ for $m > 0$ this is called Porous Medium Equation [Vázquez 2006].
Notice $\Delta \varphi(\rho) = \operatorname{div}(\varphi'(\rho) \nabla \rho)$ so $\quad U''(\rho) = \frac{\varphi'(\rho)}{\rho}$.

Consider N with positions X_i of equal masses $1/N$ and the attracting/repulsive system[1]

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \underbrace{-\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{N}\nabla W(X_i - X_j)}_{\text{Aggregation}} \underbrace{-\frac{1}{N}\nabla V(X_i)}_{\text{Confinement}}, \qquad i = 1, \cdots, N$$

The empirical distribution is defined as $\mu_t^N = \sum_{j=1}^{N} \frac{1}{N}\delta_{X_j(t)}.$

In the sense of distributions, μ^N solves the **Aggregation Equation**

$$\partial_t \mu = \mathrm{div}(\mu \nabla (W * \mu + V)) \qquad \text{(AE)}$$

Diffusion can added to the particle system by introducing noise Details .

[1] Assume $\nabla W(0) = 0$

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Joining the many particle approximation with the Porous Medium diffusion:

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\Big(\rho \nabla (U'(\rho) + V + W * \rho)\Big) \tag{ADE}$$

Some famous examples

Model	U	V	W
Heat Equation	$\rho \log \rho$	0	0
Porous Medium Equation $m \neq 1$	$\frac{1}{m-1}\rho^m$	0	0
Fokker-Planck	$\rho \log \rho$	$\frac{1}{2}\lvert x\rvert^2$	0
Keller-Segel ($d = 2$)	$\rho \log \rho$	0	$-\frac{1}{2\pi}\log\lvert x\rvert$
Swarming / Herding	0	0	$\frac{1}{a}\lvert x\rvert^a - \frac{1}{b}\lvert x\rvert^b$

In conservation laws, we expect $\int_{\mathbb{R}^d} \rho(t) = \int_{\mathbb{R}^d} \rho_0$
(i.e. $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$), we expect $\rho(t) \in \mathcal{P}(\mathbb{R}^d)$)

A direct computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \lim_{R \to \infty} \int_{B_R} \rho \, \mathrm{d}x = \lim_{R \to \infty} \int_{\partial B_R} j \frac{x}{|x|} \, \mathrm{d}S \stackrel{?}{=} 0.$$

Sometimes mass is not conserved, and we will give an example later.

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It admits a solution

$$K(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

Notice that $K(t,\cdot) \to \delta_0$ as $t \to 0^+$.

Also, for any x, $K(t,x) \to 0$ as $t \to \infty$.

We have

$$\|K(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1,$$
$$\|K(t,\cdot)\|_{L^p(\mathbb{R}^d)} \to 0, \qquad \text{as } t \to \infty \text{ for any } p > 1.$$

Furthermore, any solution satisfies

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0 \qquad \text{as } t \to \infty.$$

This is known as **asymptotic simplication**.

The range $m \in \left((\frac{d-2}{d})_+, 1 \right) \cup (1, +\infty)$:

Admits the Barenblatt solution (see [Vázquez 2006])

$$B(t,x) = t^{-\alpha} \left(C_1 - C_2 |x|^2 t^{\frac{2\alpha}{d}} \right)_+^{\frac{1}{m-1}}, \qquad \alpha = \frac{d}{d(m-1)-2}$$

That has the expect properties:

- $B(0^+, x) = \delta_0(x)$,
- $B(t,x) \to 0$ as $t \to \infty$,
- $\|B(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$, and
- The intermediate asymptotics have **asymptotic simplication**

$$\|\rho(t,\cdot) - B(t,\cdot)\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } t \to \infty.$$

For $0 < m < \frac{d-2}{d}$ there are two surprising facts:

- There is **finite-time extinction**

  If $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$ ▸ Details

  $$\|\rho(t)\|_{L^q} \searrow 0, \qquad \text{as } t \nearrow T^* < \infty.$$

- There is always **infinite-time** total mass loss.

  Take $\varepsilon > 0$, taking $\rho_0^\varepsilon \in L^q$ with $\|\rho_0 - \rho_0^\varepsilon\|_{L^1} \leq \varepsilon$. For $t \geq T_\varepsilon^*$

  $$\begin{aligned} \|\rho(t)\|_{L^1} &\leq \|\rho(t) - \rho^\varepsilon(t)\|_{L^1} + \|\rho^\varepsilon(t)\|_{L^1} \\ &\leq \|\rho_0 - \rho_0^\varepsilon\|_{L^1} + 0 \\ &\leq \varepsilon. \end{aligned}$$

- [Brezis and Friedman 1983] proved that δ_0 does not diffuse:
  if we take a sequence $\rho_j(0^+, \cdot) \to \delta_0$, the associated solution $\rho_j(t) \to \delta_0$.

Notice that the heat kernel is **self-similar**

$$K(t,x) = (2t)^{-\frac{d}{2}} G(-\tfrac{x}{\sqrt{2t}}), \qquad G(y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp(-\tfrac{|y|^2}{2})$$

So send K to G: $\tau = \log\sqrt{2t+1}$, $y = \frac{x}{\sqrt{2t+1}}$ and $u(\tau, y) = e^{d\tau} \rho\,(t,x)$

Applying the change of variable to the heat equation we recover the **Fokker-Planck equation**

$$\begin{aligned} \partial_\tau u &= \Delta_y u + \operatorname{div}(yu) \\ &= \operatorname{div}(u \nabla_y (\log u + \tfrac{|y|^2}{2})) \end{aligned}$$

Clearly, G is a stationary solution.
Furthermore, G is an **asymptotic profile** for the equation:

$$\|u(t,\cdot) - G\|_{L^1} \to 0, \qquad \text{as } t \to \infty.$$

A similar approach works for the Porous Medium Equation, where the profile is B.

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The Keller-Segel proposed a model of cell migration by chemotaxis given by

$$\begin{cases} \partial_t \rho = \Delta \rho + \operatorname{div}(\rho \nabla v), \\ -\Delta v = u. \end{cases} \qquad M = \int_{\mathbb{R}^d} \rho_0(x) \, \mathrm{d}x$$

For $d \geq 2$ we can write $v = W * u$ for W the Newtonian potential.

There exists $M^* > 0$ such that

- If $M < M^*$ solutions are global-in-time.
- If $M > M^*$ there is finite-time blow-up. And $\rho(T^*) = M\delta_0$.

The Keller-Segel proposed a model of cell migration by chemotaxis given by

$$\begin{cases} \partial_t \rho = \Delta\rho + \operatorname{div}(\rho\nabla v), \\ -\Delta v = u. \end{cases} \qquad M = \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x$$

Case $d = 2$

- [Jäger and Luckhaus 1992] There exists M^* s.t. $M > M^*$, then $\rho(T^*)$ contains a Dirac delta. They work in Ω bounded (+no-flux condition).
- [Herrero and Velázquez 1996]: For $\mathbb{R}^d$ we have $M^* = 8\pi$. Explicit constructions.
- [Dolbeault and Perthame 2004]: $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2 \rho(t,x)\,\mathrm{d}x = 4M\left(1 - \frac{M}{8\pi}\right)$
  If $M > M^*$, then $\rho(T^*) = M\delta_0$
- [Blanchet, Dolbeault, and Perthame 2006]: For $M < M^*$ global existence.

Case $d \geq 3$

- [Blanchet, Carrillo, and Laurençot 2009]:
  There is a functional $\mathcal{F}$ decaying along trajectories and
  $$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2 \rho(t,x)\,\mathrm{d}x = 2(d-2)\mathcal{F}[\rho(t,\cdot)] \leq 2(d-2)\mathcal{F}[\rho_0].$$
  When $M > M^*$ there exist $\|\rho_0\|_{L^1} = M$ such that $\mathcal{F}[\rho_0] < 0$.

For

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla \left(U'(\rho) + V + W * \rho\right)\right) \tag{ADE}$$

can we classify characterise ρ_∞ such that

$$\rho(t) \to \rho_\infty$$

in terms of general U, V, W?

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Gradient flow in $\mathbb{R}^d$

Let $F : \mathbb{R}^d \to \mathbb{R}$. Imagine we look for $\operatorname{argmin} F$.

We call **gradient flow** of F the flow of the ODE $\quad \dfrac{\mathrm{d}X}{\mathrm{d}t} = -\nabla F(X(t))$

If F is strictly convex, for any $X(0)$ we have $X(t) \to X_\infty = \operatorname{argmin} F$.

If $D^2 F \geq \lambda I$ then $|X(t) - X_\infty| \leq e^{-\lambda t}|X_0 - X_\infty|$.

The Heat Equation as a gradient flow in L^2

Let $\mathcal{F} : H^1(\mathbb{R}^d) \to \mathbb{R}$ be defined as $\mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\rho|^2$

Formally

$$\nabla_{L^2}\mathcal{F}[\rho_0] = \frac{\delta\mathcal{F}}{\delta\rho}[\rho_0] = -\Delta\rho_0$$

Remark

We can rewrite the Heat Equation

$$\frac{\partial\rho}{\partial t} = -\nabla_{L^2}\mathcal{F}[\rho(t)], \quad \text{where } \mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\rho|^2 \qquad \text{(HE)}$$

$\mathcal{F}$ is strictly convex in $L^2(\mathbb{R}^d)$. Naturally, $\rho(t) \to 0$ which is the “minimiser” of $\mathcal{F}$.

In general, the $\nabla_{L^2}\mathcal{F}$ is given by the Euler-Lagrange equations ▸ Details

Our equations are "nice" in 2-Wasserstein space ($\mathcal{P}_2$). ▸ Details
For $\mathcal{F} : L^1 \cap \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ formally speaking ▸ Details

$$\nabla_{d_2} \mathcal{F}[\rho] = -\operatorname{div}\left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho}\right)$$

Remark

If $W(x) = W(-x)$, we can formally rewrite the Aggregation-Diffusion problem as[2]

$$\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)], \quad \text{where } \mathcal{F}[\rho] = \int_{\mathbb{R}^d} \left(U(\rho) + V\rho + \tfrac{1}{2}\rho(W * \rho) \right). \qquad \text{(ADE)}$$

Formally, $\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^d} \rho \left| \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho] \right|^2$. This is called *energy dissipation* estimate.

The precise definition of solution is the notion of *curves of maximal slope*. ▸ Details

[2] Due to the convolution, $\mathcal{F}$ is non-local and $\mathcal{F}[\rho] \neq \int_{\mathbb{R}^d} F(x, \rho(x))\, \mathrm{d}x$. $\frac{\delta F}{\delta \rho}$ can be computed directly

The extension of convexity in $\mathbb{R}^d$ that is suitable in $\mathcal{P}_2$ is **displacement convexity** (see [McCann 1997])

There is a suitable theory for gradient flow of $\mathcal{F}$ in d_2 (see [Ambrosio, Gigli, and Savare 2005])

In fact, as $t \to \infty$ we have

$$\mathcal{F}[\rho(t)] \searrow \inf_{\rho \in \mathcal{P}_2 \cap L^1} \mathcal{F}.$$

Under stronger hypothesis, if $\rho_\infty = \operatorname{argmin} \mathcal{F}$ then

$$d_p(\rho(t), \rho_\infty) \to 0.$$

Due to the estimate above, at an minimiser ρ_∞, we have

$$\rho_\infty \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = 0.$$

Either $\rho_\infty = 0$ (as in PME), or $\frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = C$ (over open sets).

Minimisation and (ADE)

The free energy for (ADE)

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} \left(U(\rho) + V\rho + \frac{1}{2}\rho(W * \rho) \right) .$$

In some cases it is displacement convex (see [Carrillo and Slepčev 2009]):
$m \geq \frac{d-1}{d}$ and V, W convex.

Recall for $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ we can leave $\mathcal{P}_2$.

When $\inf \mathcal{F} = -\infty$, then we do not expect an asymptotic equilibrium.
(maybe intermediate asymptotics)

Actually, we need to consider the extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$, which we denote $\widetilde{\mathcal{F}}$
(see [Demengel and Temam 1986]) ▸ Details

If $\mu_\infty \in \operatorname{argmin}_{\|\mu\|=M} \widetilde{\mathcal{F}}[\mu]$, we expect it to be a local attractor (no guarantee).

The first variation is: $\frac{\delta \mathcal{F}}{\delta \rho} = U'(\rho) + V + W * \rho.$

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The free energy for Keller-Segel is

$$\mathcal{F}[\rho] = \underbrace{\int_{\mathbb{R}^2} \rho \log \rho}_{\mathcal{S}[\rho]} + \underbrace{\frac{-1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho(x)\rho(y) \log|x-y| \, \mathrm{d}x \, \mathrm{d}y}_{\mathcal{I}[\rho]} .$$

To see whether diffused or concentrated is energy beneficial: $\rho_\lambda(x) = \lambda^2 \rho_1(\lambda x)$.

Then, letting $M = \int_{\mathbb{R}^2} \rho_1$ we have

$$\mathcal{S}[\rho_\lambda] = \mathcal{S}[\rho_1] + 2M \log \lambda$$

$$\mathcal{I}[\rho_\lambda] = \mathcal{I}[\rho_1] - \frac{M^2}{2\pi} \log \lambda$$

Eventually

$$\mathcal{F}[\rho_\lambda] = \mathcal{F}[\rho_1] + 2M \log \lambda \left(1 - \frac{M}{8\pi} \right)$$

This gives the intuition that for $M > 8\pi$ then δ_0 (i.e. $\lambda \to \infty$) is energy beneficial.
If $M < 8\pi$ then fully diffused (i.e. $\lambda \to 0$) is beneficial.

Free-energy minimisation for (ADE) when $V = 0$

Existence and non-existence of δ_0

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Minimisation for $U = \frac{m}{m-1}\rho^m$, $V = 0$, and $W(x) = |x|^\lambda/\lambda$:

- [Carrillo, Hittmeir, Volzone, and Yao 2019]:
  Any minimiser is $\mu_\infty = \rho_\infty + M\delta_0$ with ρ_∞ radially symmetric
- [Carrillo, Delgadino, Dolbeault, Frank, and Hoffmann 2019]:
  $\lambda > 0$ and $m \in (0, 1)$
  - $\mathcal{F}$ is bounded below iff $m \in (\frac{d}{d+\lambda}, 1)$
  - If $m > \frac{d}{d+\lambda}$ there exists a minimiser of the form $\mu_\infty = \rho_\infty + M\delta_0$
  - If $\lambda \in [2, 4]$ or $\lambda \geq 1$ and $m \geq 1 - \frac{1}{d}$, then the global minimiser is unique (up to translation).
  - $M = 0$ if $\lambda \in (0, 2 + \frac{4}{(N-2)_+})$ and $m \in (\frac{d}{d+\lambda}, 1)$
- [Carrillo, Delgadino, Frank, and Lewin 2020]:
  - If $\lambda = 4$ and $d \leq 5$ then $M = 0$.
  - If $\lambda = 4$ and $d \geq 6$ then $M = 0$ if and only if $m \geq \frac{d-2}{d+4}\left(1 + \frac{4}{3d}\right)$.
  - Numerical results for $\lambda = 2k$

Asymptotic behaviour as $t \to \infty$

- [Cañizo, Carrillo, and Schonbek 2012]: for small W: $\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0$, like (HE).
- [Carrillo, G-C, Yao, and Zeng 2021]:
  $W \in \mathcal{W}^{1,\infty}$, $\nabla W \in L^{n-\varepsilon}$, $\Delta W \in L^{\frac{n}{2}-\varepsilon}$ then $\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0$, like (HE) .

We start by a classical observation.

Remark

If $W \in L^\infty(\mathbb{R}^d)$, then there are no finite-mass steady states.

If $W \in L^\infty(\mathbb{R}^d)$ and $\rho \in L^1(\mathbb{R}^d)$, then $W * \rho \in L^\infty(\mathbb{R}^d)$.
The Euler-Lagrange equation is $\log \rho + W * \rho = c$, so

$$\rho = e^c e^{W*\rho} \geq e^c e^{-\|W*\rho\|_{L^\infty}} > 0.$$

Therefore, in this range we always expect diffusion.

In the case of linear diffusion

$$\partial_t \rho = \Delta \rho + \mathrm{div}(\rho \nabla W * \rho).$$

[Cañizo, Carrillo, and Schonbek 2012]: for small W:

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0 \qquad \text{as } t \to \infty. \tag{$\star$}$$

where K is the heat kernel.

Theorem [Carrillo, G-C, Yao, and Zeng 2021]

Let $n \geq 2$, and assume $W(x) = W(-x)$

- $W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$
- $\nabla W \in L^{n-\varepsilon}(\mathbb{R}^d)$
- $\Delta W \in L^{\frac{n}{2}}(\mathbb{R}^d)$ (and also $\Delta W \in L^{\frac{n}{2}-\varepsilon}(\mathbb{R}^d)$ if $n \geq 3$)

Then ($\star$).

Notice these holds for $W(x) \sim |x|^{-\varepsilon}$ for any $\varepsilon > 0$, but not $W(x) \sim \log|x|$.
More to come in Yao's talk.

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Minimisation for (ADE) when $m \in (0,1)$, $W = 0$ and V radially non-decreasing

As presented in [Carrillo, G-C, and Vázquez 2021]

We restrict ourselves to (ADE) on a ball B_R (with no-flux conditions).

$$\mathcal{F}[\rho] = \frac{1}{m-1} \int_{B_R} \rho^m + \int_{B_R} V \rho.$$

The energy is bounded below because $\int_{B_R} \rho^m \le (\int_{B_R} \rho)^m |B_R|^{1-m}$.
The case $\mathbb{R}^d$ is similar, taking care of the tails.

The proof of the energy minimisation goes as follows

1. Take a minimising sequence ρ_k s.t. $\|\rho_k\|_{L^1} = \|\rho_0\|_{L^1}$.
2. Use symmetrisation to change to a radially symmetric minimising sequence, still denote ρ_k.
3. $\rho_k \overset{\star}{\rightharpoonup} \mu_\infty = \rho_\infty \, \mathrm{d}x + M\delta_0$ (up to subsequence)
   - 3.1 Use tightness of measures to pass to a limit $\rho_k \overset{\star}{\rightharpoonup} \mu_\infty$ in $M(B_R)$.
   - 3.2 Using ▸ Lieb's trick to have a universal bound in $L^\infty_{loc}(\mathbb{R}^d \setminus \{0\})$. The singular part is $M\delta_0$
4. By weak lower semi-continuity (e.g. [Demengel and Temam 1986]), μ_∞ is a minimiser of

$$\widetilde{\mathcal{F}}[\mu] = \tfrac{1}{m-1} \int_{B_R} (\mu_{ac})^m + \int_{B_R} V \, \mathrm{d}\mu$$

5. $\rho_\infty = \rho_{V+h}\chi_{B_{R_\infty}}$ for some $h \ge 0$ and R_∞ where $\rho_{V+h} = \left(\frac{1-m}{m}(V+h)\right)^{-\frac{1}{1-m}}$

   5.1 Notice that $\widetilde{\mathcal{F}}[\mu_\infty] = \mathcal{F}[\rho_\infty]$.

   5.2 Take $\varphi \in C_c^\infty(\mathbb{R}^d)$ and

$$\psi(x) = \left(\varphi(x)\int_{B_R}\rho_\infty(y)\,\mathrm{d}y - \int_{B_R}\varphi(y)\rho_\infty(y)\,\mathrm{d}y\right)\rho_\infty(x).$$

   5.3 Since $\mathcal{F}[\rho_\infty] \le \mathcal{F}[\rho_\infty \pm \varepsilon\psi]$, we deduce, defining $I[\rho] := \frac{m}{m-1}\rho^{m-1} + V$ that

$$\int_{B_R} I[\rho_\infty](x)\psi(x) = 0.$$

   Thus

$$\rho_\infty(x) = 0 \qquad \text{or} \qquad I[\rho_\infty](x) = \underbrace{\frac{\int_{B_R} I[\rho_\infty](y)\rho_\infty(y)\,\mathrm{d}y}{\int_{B_R}\rho_\infty(y)\,\mathrm{d}y}}_{\text{constant}}.$$

   5.4 Recall that ρ_∞ is radially decreasing, so when it reaches 0 it stays at 0

6. $\mu_\infty = (1 - \mathfrak{m}_h)\delta_0 + \left(\frac{1-m}{m}(V+h)\right)^{-\frac{1}{1-m}} \, \mathrm{d}x$,
with $h \geq 0$ as small as possible and $\mathfrak{m}_h = \int_{B_R} \left(\frac{1-m}{m}(V+h)\right)^{-\frac{1}{1-m}}$.
Direct optimisation in (h, R_∞).

Thus,

$$\mu_\infty = \begin{cases} \rho_{V+h} & \text{if } \exists h \geq 0 \text{ such that } \|\rho_{V+h}\|_{L^1} = \|\rho_0\|_{L^1}, \\ \rho_V + \Big(\|\rho_0\|_{L^1} - \|\rho_V\|_{L^1} \Big) \delta_0 & \text{if } \|\rho_0\|_{L^1} > \|\rho_V\|_{L^1} \end{cases}$$

The main question of the paper is: let ρ_t be the solution of (ADE):

$$\rho_t \xrightarrow{?} \mu_\infty \qquad \text{as } t \to \infty$$

We have shown that the minimiser of the free energy is

$$\mu_\infty = \begin{cases} \rho_{V+h} & \text{if } \exists h \ge 0 \text{ such that } \|\rho_{V+h}\|_{L^1} = \|\rho_0\|_{L^1}, \\ \rho_V + \Big(\|\rho_0\|_{L^1} - \|\rho_V\|_{L^1}\Big)\delta_0 & \text{if } \|\rho_0\|_{L^1} > \|\rho_V\|_{L^1} \end{cases}$$

[Cao and Li 2020]: If $\rho_{V+h_1} \le \rho_0 \le \rho_{V+h_2}$ with $h_1, h_2 > 0$ then positive answer.

[Carrillo, G-C, and Vázquez 2021]: if $V \in W^{2,\infty}(B_R)$
there exist ρ_0 such that $\|\rho_0\|_{L^1} > \|\rho_V\|_{L^1}$ and the result holds.

Main results in [Carrillo, G-C, and Vázquez 2021]

Theorem 1

Assume $V \in W^{2,\infty}(B_R)$ is radially symmetric, strictly increasing, $V \geq 0$, $V(0) = 0$, $V \cdot x = 0$ on ∂B_R

$$\int_{B_1} \rho_V^{1+\varepsilon}(x)\,\mathrm{d}x < +\infty, \qquad \text{for some } \varepsilon > 0. \tag{1}$$

Assume also that $a_{0,R} = \int_{B_R} \rho_0 > \int_{B_R} \rho_V = a_{V,R}$,
ρ_0 radially symmetric, $\rho_0 \geq \rho_V$ and $\rho_0 \in L^\infty(B_R \setminus B_{r_1})$ for some $r_1 < R$.
Then, the solution ρ satisfies

$$\liminf_{t\to\infty} \int_{B_r} \rho(t,x)\,\mathrm{d}x \geq (a_{0,R} - a_{V,R}) + \int_{B_r} \rho_V(x)\,\mathrm{d}x, \qquad \forall r \in [0,R].$$

(i.e., there is concentration in infinite time).
Moreover, if

$$\int_{B_r} \rho_0(x)\,\mathrm{d}x \leq (a_{0,R} - a_{V,R}) + \int_{B_r} \rho_V(x)\,\mathrm{d}x \qquad \forall r \in [0,R], \tag{2}$$

then for $\mu_{\infty,R} = (a_{0,R} - a_{V,R})\delta_0 + \rho_V$ we have that

$$\lim_{t\to\infty} d_1(\rho(t), \mu_{\infty,R}) = 0,$$

where d_1 denotes the 1-Wasserstein distance.

Corollary 2 At least infinite-time concentration of solutions

Under the hypothesis of Theorem 1 and suitable hypothesis on the initial data, we can show the existence of viscosity solutions of $(\star)$ in $(0,\infty)\times(0,\infty)$ (obtained as a limit of the problems in B_R), such that

$$\lim_{t\to\infty} M(t,v) = (1-a_V) + M_{\rho_V}(v)$$

for all $v>0$ and, furthermore, locally uniformly $(0,\infty)$. We also have that

$$\lim_{t\to\infty} d_1(\rho(t),(1-a_V)\delta_0+\rho_V) = 0.$$

1. Existence by approximation as in [Vázquez 2006].
   Classical results guarantee existence of $\partial_t u = \Delta\varphi(u) + \operatorname{div}(u\nabla V)$ with φ and V regular.

   For technical convenience, we work in a ball B_R, $\nabla u \cdot n = \nabla V \cdot n = 0$ on ∂B_R.

2. Analysis of the mass function for radial solution $M(t,r) = \int_{B_r} \rho(t,x)\,\mathrm{d}x$.
   Taking volume variable $v = \omega_n r^d$ (then $\frac{\partial M}{\partial v} = \rho$), M satisfies

$$\frac{\partial M}{\partial t} = (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ \frac{\partial}{\partial v}\left[\left(\frac{\partial M}{\partial v}\right)^m\right] + \frac{\partial M}{\partial v}\frac{\partial V}{\partial v} \right\} \qquad (\star)$$

   We develop a theory of viscosity solutions with comparison principle.

3. To check the limit $t \to \infty$ we consider
   $M_n(t,v) = M(t+n,v)$ as $[0,1] \times B_R$ functions.
4. By C^α away from 0 (see [DiBenedetto 1993]),
   $M_{n_k} \to M_\infty$ uniformly over compacts of $[0,1] \times (0,R]$.
5. By stability of viscosity solutions M_∞ is a solution of $(\star)$.

6. To check that $M_\infty = M_\infty(x)$ (i.e. it does not depend on t), **we construct initial data** ρ_0 so that $\partial M/\partial t \geq 0$.
   - Let $U = \frac{\partial M}{\partial t}$
   - Taking a derivative in $(\star)$ we observe that if $U(0,v) \geq 0$ then $U(t,v) \geq 0$.
   - We notice that $(\star)$ at $t = 0$ says

$$U(0,v) = (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ \frac{\partial}{\partial v}\left[\left(\frac{\partial M_0}{\partial v}\right)^m\right] + \frac{\partial M_0}{\partial v}\frac{\partial V}{\partial v} \right\}$$

   - We can pick $\rho_0(x) = \left(\frac{1-m}{m}F(V(x))\right)^{-\frac{1}{1-m}}$, with $F' \leq 1$, $F(0) = 0$, $F(s) > 0$ for all $s > 0$.
   - These are formal computations, that are rigorously justified in the approximation scheme. The fact that M is non-decreasing in t stays.
7. By uniform C^α, M_∞ is continuous, and by Dini's theorem $M_n \nearrow M_\infty$ on $[0,T] \times (0,R]$ uniformly over compacts.
8. So M_∞ is a viscosity solution of $\frac{\partial}{\partial v}\left[\left(\frac{\partial M}{\partial v}\right)^m\right] + \frac{\partial M}{\partial v}\frac{\partial V}{\partial v} = 0$ and $M_\infty(R) = \|\rho_0\|_{L^1(B_R)}$.
9. Then $M_\infty(r) = \mu_\infty(B_r)$.
10. By comparison, this works for any initial datum above the class constructed.

Through approximation as $R \to \infty$, we will also show that

Corollary 3 At least infinite-time concentration of solutions

Under the hypothesis of Theorem 1 and suitable hypothesis on the initial data, we can show the existence of viscosity solutions of ($\star$) in $(0,\infty) \times (0,\infty)$
(obtained as a limit of the problems in B_R), such that

$$\lim_{t\to\infty} M(t,v) = (1 - a_V) + M_{\rho_V}(v)$$

for all $v > 0$ and, furthermore, locally uniformly $(0,\infty)$. We also have that

$$\lim_{t\to\infty} d_1(\rho(t), (1 - a_V)\delta_0 + \rho_V) = 0.$$

Through our construction of M, we cannot guarantee in general that $M(t,0) = 0$ for t finite.
Producing a priori estimates, we can ensure this in some cases.

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Theorem 4 Infinite-time concentration for V quadratic at 0

Let $\rho_0 \in L^1_+(\mathbb{R}^n)$ non-increasing and assume

$$\frac{\partial V}{\partial r}(s) \leq C_V r, \qquad \text{in } B_{R_V} \text{ for some } C_V > 0. \tag{3}$$

Then, the viscosity mass solution does not concentrate in finite time, i.e. $M(t, 0) = 0$.

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Question that arise:

1. Does this idea work for any $\rho_0 \in L^1$?
2. Does it work for some classes $W \neq 0$.

The answer seems positive for some W.
PhD plan of Alejandro Fernández-Jiménez (U. Oxford).

The difficulty is proving that M_∞ does not depend on t.
$M_n \nearrow$ should replaced by a compactness argument.

Thank you for your attention!

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Consider an stochastic particle jumping over the mesh $\{\dots, -h, 0, h, 2h, \dots\}$ $(h > 0)$.
Let X_n be the position after n jumps. Assume the jump probabilities are

$$\mathbb{P}(X_{n+1} = jh \mid X_n = ih) = \begin{cases} \frac{1}{2} & \text{if } |i - j| = 1, \\ 0 & \text{otherwise} \end{cases}$$

Define $U_j^n = P(X_n = hj)$ and assume the initial distribution U_j^0 is given.
Then $U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n)$ or, for $\tau = h^2$

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \frac{1}{2} \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}.$$

This is a classical discretisation of the stochastic version of (HE): $\partial_t \rho = \frac{1}{2}\Delta\rho$.

The time continuous extension of X_n version is the Wiener process $X_t = W_t$.
This gives rise to the intuition (which has to be understood in terms of the Îto calculus)

$$dX_t = dW_t$$

Consider 1 particle. Using a similar arguments, for the stochastic equation

$$\mathrm{d}X_t = \underbrace{\mu(t, X_t)\,\mathrm{d}t}_{\text{drift}} + \underbrace{\sigma(t, X_t)\,\mathrm{d}W_t}_{\text{diffusion}}$$

its probability density ρ is the solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t}(t, x) = -\operatorname{div}(\mu(t, x)\rho(t, x)) + \Delta(D(t, x)\rho(t, x))$$

where $D = \frac{\sigma^2}{2}$.

Imagine now we have N stochastic particles at positions $X_1(t), \cdots, X_N(t)$. We assume they have equal mass.

Recall the empirical measure $\mu_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_j(t)}$

Assume the evolution of the particles is given by the system of SODEs

$$\mathrm{d}X_i = -\frac{1}{N} \sum_{j \neq i} \nabla W(X_i - X_j) - \frac{1}{N} \nabla V(X_i) + \sqrt{2D}\, \mathrm{d}W_t^i$$

The limit as $N \to \infty$ is the solution of

$$\partial_t \rho = \mathrm{div}(\rho \nabla (W * \rho + V)) + D \Delta \rho.$$

This corresponds to $U(\rho) = D\rho \log \rho$.

Mean-Field Approximation: As $N \to \infty$

$$\mu_0^N \to \rho_0 \text{ in the tight topology} \implies \mu_t^N \to \rho(t) \text{ in law for a.e. } t > 0.$$

For the details see, e.g., [Jabin and Wang 2017].

[2]Convergence in law: pointwise convergence of distribution functions at continuity points of the limit

Let $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ and $d \geq 3$ and $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{q} \int_{\mathbb{R}^d} \rho^q \overset{\text{PDE}}{=} -C \int_{\mathbb{R}^d} |\nabla \rho^{\frac{m+q}{2}}|^2 \overset{\text{Sobolev}}{\leq} -C \left(\int_{\mathbb{R}^d} \rho^{\frac{m+q}{2} 2^*} \right)^{\frac{1}{2^*}}$$

where $2^* = \frac{1}{2} - \frac{1}{d}$.
The equation $\frac{\mathrm{d}}{\mathrm{d}t} X = -C X^\alpha$ where $\alpha < 1$ has finite time extinction.

Computation of the Wasserstein gradient

Following [Ambrosio, Gigli, and Savare 2005, §10.4.1] Go back

$\mathcal{P}_2$ is not a vector space, so there we are not using the intrinsic notion of Fréchet gradient.
The correct notion is **Fréchet subdifferentials** (we will not define it here).
Also, we can see $\mathcal{P}_2$ inside the space of measures.
Fix $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then, the tangent space is given by

$$\operatorname{Tan}_{\rho_0}\mathcal{P}_2(\mathbb{R}^d) = \left\{ \xi : \exists \zeta_n \in C_c(\mathbb{R}^d, \mathbb{R}) \text{ s.t. } \int_{\mathbb{R}^d} |\xi - \nabla \zeta_n|^2 \, \mathrm{d}\rho_0 \to 0 \right\}$$

Take $\xi = \nabla \zeta$ with $\zeta \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Then, by [Ambrosio, Gigli, and Savare 2005, Lemma 5.5.3]

$$\rho_\varepsilon = (1_{\mathbb{R}^d} + \varepsilon \xi)_\# \rho_0 = \frac{\rho_0}{\det(1_{\mathbb{R}^d} + \varepsilon \nabla \xi)} \circ (1_{\mathbb{R}^d} + \varepsilon \xi)^{-1}$$

The map $(x, \varepsilon) \mapsto \rho_\varepsilon(x)$ is C^2 and

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = \rho_0, \qquad \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon = 0} \rho_\varepsilon = -\operatorname{div}(\rho \xi).$$

For ε small enough $1_{\mathbb{R}^d} + \varepsilon \nabla \zeta$ is an optimal transport map, so ρ_ε is a constant-speed geodesic.
Hence, using standard variation formulae (see [Giaquinta and Hildebrandt 1996])

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}[\rho_\varepsilon] - \mathcal{F}[\rho_0]}{\varepsilon} = - \int_{\mathbb{R}^d} \frac{\delta F}{\delta \rho}[\rho_0] \nabla \cdot (\rho \xi) = \int_{\mathbb{R}^d} \nabla \zeta \nabla \frac{\delta F}{\delta \rho}[\rho_0] \, \mathrm{d}\rho$$

This characterises $\nabla_{d_2} \mathcal{F} = -\operatorname{div}(\rho \nabla \frac{\delta F}{\delta \rho})$ in a broad distributional sense.

Convex functions of a measure

Following [Demengel and Temam 1986] Go back

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Given

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} f(\rho)\, \mathrm{d}x$$

where $f : \mathbb{R} \to \mathbb{R}$.

The questions is what is the natural lower semicontinuous extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$ with the weak-$\star$ topology.

Given a measure μ and mollifiers η_ε we define $\rho_\varepsilon = \mu * \eta_\varepsilon$.
For $|f(\xi)| \le C(1 + |\xi|)$ define

$$f_\infty(\xi) = \lim_{t\to\infty} \frac{f(t\xi)}{t}.$$

Since we can use the Lebesgue decomposition theorem $\mu = \rho\, \mathrm{d}x + \mu^s$, where ρ is the Radon-Nikodym derivative of μ. Then

$$\widetilde{F}[\mu] = \int_{\mathbb{R}^d} f(\rho)\, \mathrm{d}x + f_\infty(\mu^s).$$

The notion of $f_\infty(\mu^s)$ is tricky (but possible) to define.
If $f(s) = s^m$ with $m < 1$, then $f_\infty = 0$.

Curves of maximal slope

(see [Ambrosio, Gigli, and Savare 2005]) ▸ Go back

Typically, $\frac{\partial \rho}{\partial t} = -\nabla_X \mathcal{F}[\rho(t)]$ for $X = L^2, H^1$ is satisfied in the dual sense.

The way in which $\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)]$ in rather tricky since $\mathcal{P}_2$ is not linear a space.

The main idea is the equivalence for $u : [0, T] \to \mathbb{R}^d$ that

$$u'(t) = -\nabla \mathcal{F}(u), \qquad \iff \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F} \circ u) = -|\nabla F(u)||u'| & \text{orientation} \\ |u'| = |\nabla \mathcal{F}(u)| & \text{norm} \end{cases}$$

We define the metric slopes

$$|\mu'|(t) = \limsup_{h \to 0} \frac{d_2(\mu(t+h), \mu(t))}{h}, \qquad |\partial \mathcal{F}|[\mu] = \limsup_{\nu \to \mu} \frac{(\mathcal{F}[\mu] - \mathcal{F}[\nu])_+}{d_2(\mu, \nu)}$$

Definition 5 Maximal slope curve

A locally abs. cont. curve $t \mapsto \mu(t) \in \mathcal{P}_2(\mathbb{R}^d)$ such that $t \mapsto \mathcal{F}[\mu(t)]$ is abs. cont. and

$$\frac{1}{2} \int_s^t |\mu'|^2(r) \, \mathrm{d}r + \frac{1}{2} \int_s^t |\partial \mathcal{F}|^2[\mu(r)] \, \mathrm{d}r \le \mathcal{F}[\mu(s)] - \mathcal{F}[\mu(t)] \qquad \forall 0 \le s < t \le T$$

Let

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} F(x, \rho(x), \nabla\rho(x))\, \mathrm{d}x.$$

Expanding $F(x, s, \xi)$ in Taylor expansion yields

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{\mathcal{F}[\rho_0 + \varepsilon\varphi] - \mathcal{F}[\rho_0]}{\varepsilon} &= \int_{\mathbb{R}^d} \left(\tfrac{\partial F}{\partial s}(x, \rho_0, \nabla\rho_0)\varphi + \tfrac{\partial F}{\partial \xi}(x, \rho_0, \nabla\rho_0) \cdot \nabla\varphi \right) \\ &= \int_{\mathbb{R}^d} \left(\tfrac{\partial F}{\partial s}(x, \rho_0, \nabla\rho_0) - \operatorname{div}\left[\tfrac{\partial F}{\partial \xi}(x, \rho_0, \nabla\rho_0) \right] \right) \varphi \end{aligned}$$

Thus

$$\nabla_{L^2}\mathcal{F}[\rho_0] = \frac{\delta\mathcal{F}}{\delta\rho}[\rho_0] = \frac{\partial F}{\partial s}[\rho_0] - \operatorname{div}\left(\frac{\partial F}{\partial \xi}[\rho_0] \right).$$

This is the Euler-Lagrange equation!

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Go back Given a radially decreasing $\rho \geq 0$, $\rho^q \in L^1(B_R)$ for some $q > 0$ (for any $R \leq \infty$), using and old trick of Lieb's (see [Lieb 1977; Lieb 1983]) we get, for $|x| \leq R$,

$$\int_{B_R} \rho^q \, \mathrm{d}x = n\omega_n \int_0^R \rho(r)^q r^{n-1} \, \mathrm{d}r \geq n\omega_n \int_0^{|x|} \rho(r)^q r^{n-1} \, \mathrm{d}r \geq n\omega_n \rho(x)^q \int_0^{|x|} r^{n-1} \, \mathrm{d}r.$$

Hence, we deduce the point-wise estimate

$$\rho(x) \leq \left(\frac{\int_{B_R} \rho^q}{n\omega_n |x|^n} \right)^{\frac{1}{q}} . \tag{4}$$

It is easy to see that (4) is not sharp. However, it is useful to prove tightness for sets of probability measures.