Aggregation–Diffusion Equations: concentration vs simplification

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Congreso de jóvenes investigadores de la RSME,
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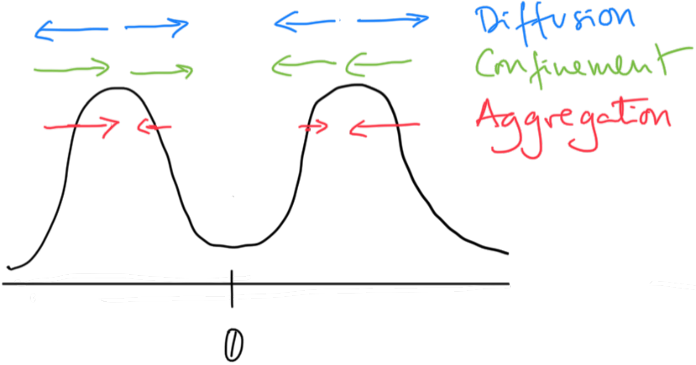
Aggregation and Diffusion



A model of diffusion, confinement and aggregation

The aim of this talk is to explain the modeling and theory behind the following model for aggregation-diffusion phenomena:

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla\big(\underbrace{U'(\rho)}_{\text{Diffusion}} + \underbrace{V}_{\text{Confinement}} + \underbrace{W * \rho}_{\text{Aggregation}} \big)\right) \qquad \text{(ADE)}$$

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We will discuss the range of power-type aggregation and diffusion

$$U'(\rho) = \tfrac{m}{m-1}\rho^{m-1}, \qquad V(x) = \frac{|x|^\alpha}{\alpha}, \qquad \text{and} \qquad W(x) = \frac{|x|^\lambda}{\lambda}.$$

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If V, W are bounded below, we can always assume $V, W \geq 0$.

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Notice $\Delta\varphi(\rho) = \operatorname{div}(\varphi'(\rho)\nabla\rho)$ so $\qquad U''(\rho) = \frac{\varphi'(\rho)}{\rho}$.

Particle systems

Consider N with positions X_i of equal masses $1/N$

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Diffusion can added to the particle system by introducing noise

▸ Details .

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The Aggregation-Diffusion Equation

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Keller-Segel ($d = 2$)	$\rho \log \rho$	0	$-\frac{1}{2\pi}\log\|x\|$
Swarming / Herding	0	0	$\frac{1}{a}\|x\|^a - \frac{1}{b}\|x\|^b$

Conservation?

In conservation laws, we expect $\int_{\mathbb{R}^d} \rho(t) = \int_{\mathbb{R}^d} \rho_0$

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Sometimes mass is not conserved, and we will give an example later.

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It admits a solution

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- Any solution satisfies

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0 \qquad \text{as } t \to \infty.$$

This is known as **asymptotic simplication**.

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- [Brezis and Friedman 1983] proved that δ_0 does not diffuse: if we take a sequence $\rho_j(0^+, \cdot) \to \delta_0$, the associated solution $\rho_j(t) \to \delta_0$.

Asymptotic profiles

Notice that the heat kernel is **self-similar**

$$K(t,x) = (2t)^{-\frac{d}{2}} G(-\tfrac{x}{\sqrt{2t}}), \qquad G(y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp(-\tfrac{|y|^2}{2})$$

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Furthermore, G is an **asymptotic profile** for the equation:

$$\|u(t,\cdot) - G\|_{L^1} \to 0, \qquad \text{as } t \to \infty.$$

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Notice that the heat kernel is **self-similar**

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So send K to G: $\tau = \log\sqrt{2t+1}$, $y = \frac{x}{\sqrt{2t+1}}$ and $u(\tau, y) = e^{d\tau}\rho(t,x)$

Applying the change of variable to the heat equation we recover the **Fokker-Planck equation**

$$\begin{aligned}\partial_\tau u &= \Delta_y u + \mathrm{div}(yu)\\ &= \mathrm{div}(u\nabla_y(\log u + \tfrac{|y|^2}{2})).\end{aligned}$$

Clearly, G is a stationary solution.
Furthermore, G is an **asymptotic profile** for the equation:

$$\|u(t,\cdot) - G\|_{L^1} \to 0, \qquad \text{as } t \to \infty.$$

A similar approach works for the Porous Medium Equation, where the profile is B.

Finite time blow-up

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$$\begin{cases} \partial_t \rho = \Delta \rho + \operatorname{div}(\rho \nabla v), \\ -\Delta v = u. \end{cases} \qquad M = \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x$$

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There exists $M^* > 0$ such that

- If $M < M^*$ solutions are global-in-time.
- If $M > M^*$ there is finite-time blow-up. And $\rho(T^*) = M\delta_0$.

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  When $M > M^*$ there exist $\|\rho_0\|_{L^1} = M$ such that $\mathcal{F}[\rho_0] < 0$.

Main question

For

$$\frac{\partial \rho}{\partial t} = \mathrm{div}\left(\rho \nabla (U'(\rho) + V + W * \rho)\right) \qquad \text{(ADE)}$$

can we classify characterise ρ_∞ such that

$$\rho(t) \to \rho_\infty$$

in terms of general U, V, W?

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The Heat Equation as a gradient flow in L^2

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In general, the $\nabla_{L^2}\mathcal{F}$ is given by the Euler-Lagrange equations

Gradient flows in Wasserstein space

Our equations are "nice" in 2-Wasserstein space ($\mathcal{P}_2$).
For $\mathcal{F} : L^1 \cap \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ formally speaking

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[2]Due to the convolution, $\mathcal{F}$ is non-local and $\mathcal{F}[\rho] \neq \int_{\mathbb{R}^d} F(x, \rho(x))\, \mathrm{d}x$. $\frac{\delta F}{\delta \rho}$ can be computed directly

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This is called *energy dissipation* estimate.

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The precise definition of solution is the notion of *curves of maximal slope*.

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$$\mathcal{F}[\rho(t)] \searrow \inf_{\rho \in \mathcal{P}_2 \cap L^1} \mathcal{F}.$$

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Gradient-flow structure and minimisation

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Due to the energy dissipation a minimiser ρ_∞ should satisfy

$$\rho_\infty \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = 0.$$

Either $\rho_\infty = 0$ (as in PME), or $\frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = C$ (over open sets).

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This gives the intuition that for $M > 8\pi$ then δ_0 (i.e. $\lambda \to \infty$) is energy beneficial.

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We go back to

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$$\rho(h) = \left(\tfrac{1-m}{m} (V + h) \right)^{\frac{-1}{1-m}}$$

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We are able to prove, in some cases, that they are global attractors of (ADE): talk of A. Fernández-Jiménez.

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Therefore, in this range we always expect diffusion.

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Then $(\star)$.

Notice that this hypothesis work for $W(x) \sim |x|^{-\varepsilon}$ for any $\varepsilon > 0$, but not for the critical case $W(x) \sim \log|x|$.

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- Thus, using the control of $\int \rho \log \rho$ we prove that

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$\square$

Questions, comments, remarks, ...

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